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Comparisons of Various Root-finding Methods Based on Their Basins of Attraction

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Abstract
The primary discussion of this study is to compare various root-finding methods based on their basins of attraction. The studied methods are taken from various order of convergence and efficiency index. We consider number of divergent points to make clear of the observations on the behavior of the studied methods. The relationship of the order of convergence and the efficiency index to the basins of attraction is studied.
Keywords : nonlinear equations, iterations methods, efficiency index, order of convergence, basins of attraction

1. Introduction

Nonlinear equations which is expressed by

$$f(x) = 0 \tag{1}$$

has been playing an important role in mathematics and its applications. Equations (1) has been used to model problems in chemistry, economics, marketing, physics, and more. The solution(s) of (1) can be found analytically and or numerically depending on the nature of the functions. Numerical methods to find solutions of (1) is one of the popular topics in mathematics.

Root-finding methods are classified by their order of convergence, p, and the number of function evaluations per step, d. To measure the efficiency of the methods, Traub [1] introduces informational efficiency which is a defined as $I = p/d$ and efficiency index which is given by $p^{1/d}$. Another measure introduced recently is basin of attraction, see [2], [3], [4], [5], [6].

Researchers have developed efficient algorithms to approximate the solution of nonlinear equations. Such algorithm is preferred to be higher order since it has shown that methods with this kind tend to converge faster to the roots. However, this is not always the case if the efficiency index of the method is compromised by the number of function evaluations at each step. Then the method will no longer be favorable or considered efficient.

In this study, we are interested in comparing various root-finding methods and measure the efficiency based on the basins of the attractions of the methods. The methods that are considered are higher order methods. In the following section, we outline the methods that should be examined in our comparative analysis where we observe the basins of attraction and the number of converged points for each studied method. The conclusion will be given in Section 3.

2. Iterative Methods for Basins of Attraction Comparative Analysis

In this section, we display several higher order iterative methods of different kinds. We highlight the order of convergence and efficiency index to be the comparative factors of the methods to further do the analysis on the basins of attraction in the following section.

We list four methods with their order of convergence. The first method is a method of order 36th by Ahmad, et al. [7]. This method has 10 function evaluation for each step, so its efficiency index is

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$36^{1/10} \approx 1.431$. The method is given by the following scheme:

$$\left.\begin{aligned} a_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ b_n &= a_n - \frac{2f(a_n)f'(a_n)}{2f'(a_n)^2 - f(a_n)f''(a_n)}, \\ c_n &= b_n - \frac{f(b_n)}{f'(b_n)} - \frac{f(b_n)^2 f''(b_n)}{2(f'(b_n))^3}, \\ x_{n+1} &= c_n - \frac{f(c_n)^2}{f\big(c_n + f(c_n)\big) - f(c_n)}, \end{aligned}\right\} \tag{2}$$

for $n = 0, 1, 2, \cdots$. The method described by (2) will be referred to as AM36 for the rest of this article.

The second method is proposed by Fiza, et. al [8] which is of order 14th with five function evaluations, hence its efficiency index is $14^{1/5} \approx 1.695$. The method is described as follows:

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)} - \frac{f(w_n)}{f'(w_n)}, \end{aligned}\right\} \tag{3}$$

where

$$\begin{aligned} w_n &= z_n \frac{f(z_n)}{f'(z_n)}, \\ f'(z_n) &= f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n), \\ f'(w_n) &= f[x_n, w_n] + \big(f[y_n, x_n, z_n] - f[y_n, x_n, w_n] - f[z_n, x_n, w_n]\big)(x_n - w_n) \end{aligned}$$

The method given in (3) is referred to as FM14 in this article.

The next method is of order $9th$ with efficiency index $= 9^{1/5} \approx 1.552$ established by Noor, et.al [9]. This method will be labeled as NM9 and is presented as follows:

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{2f(y_n)f'(y_n)}{2f'(y_n)^2 - f(y_n)P_f(x_n, y_n)}, \\ x_{n+1} &= z_n - \frac{f'(x_n) + f'(y_n)}{3f'(y_n) - f'(x_n)} \frac{f(z_n)}{f'(x_n)}, \end{aligned}\right\} \tag{4}$$

with

$$P_f(x_n, y_n) = \frac{2}{y_n - x_n}\left(2f'(y_n) + f'(x_n) - 3\frac{f(y_n) - f(x_n)}{y_n - x_n}\right)$$

The ultimate method is an 8th order method developed by Al-Subaihi, et.al [10]. This method has efficiency index $= 8^{1/4} \approx 1.6818$ and will be called ASM8 for the rest of this article. The scheme of the method is given below.

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n + \frac{f(y_n)}{f'(y_n)} - 2\frac{f(x_n)f(y_n)}{f'(x_n)\big(f(x_n) - f(y_n)\big)}, \\ x_{n+1} &= y_n + c(f(x_n)^2) - d(f(x_n)^3), \end{aligned}\right\} \tag{5}$$

where

$$c = \frac{1}{\big(f(y_n)-f(x_n)\big)f[x_n,y_n]} - \frac{1}{f'(x_n)\big(f(y_n)-f(x_n)\big)} - d\big(f(y_n)-f(x_n)\big),$$

$$d = \frac{1}{\big(f(y_n)-f(x_n)\big)\big(f(y_n)-f(z_n)\big)f[y_n,x_n]} - \frac{1}{\big(f(z_n)-f(x_n)\big)\big(f(y_n)-f(z_n)\big)f[z_n,x_n]}$$
$$+ \frac{1}{f'(x_n)\big(f(z_n)-f(x_n)\big)\big(f(y_n)-f(z_n)\big)} - \frac{1}{f'(x_n)\big(f(y_n)-f(x_n)\big)\big(f(y_n)-f(z_n)\big)}$$

2.1. Basins of Attraction

In this part, we observe the behavior of the mentioned methods to solve $f(x)=0$, where $f:\mathbf{C}\longrightarrow\mathbf{C}$ and $\mathbf{C}$ is a complex plane, through their basins of attraction. We employed Maple 2023 as our tool with 50 significant digits and error tolerance as 10^{-14}. In this study, we use the following four nonlinear equations.

(1) $f_1(x)=x^3-1$, $x=\{-0.5-0.866i,-0.5+0.866i,1\}$.
(2) $f_2(x)=x^3-x$, $x=\{-1.000,0,1.000\}$.
(3) $f_3(x)=x^4-10x^2+9$, $x=\{-3,-1,1,3\}$
(4) $f_4(x)=x^5-1$, $x=\{-0.8090-0.5878i,-0.809+0.5878i,0.309-0.951i,-0.309+0.951i,0.309+0.951i,1.000\}$

In many root-finding method simulations, one usually uses initial guesses that are close enough to the actual root(s). This resulting in the limitation of the observation of the said method on other potential initial guesses. In this paper, the observation is carried by considering a large number of data as our initial guesses for each tested function. The data is constructed from a grid of $[-1,1]\times[-1,1]\subset\mathbf{C}$. Therefore we have 1000000 initial guesses for each function. For each studied method we allow 100 iterations.

In addition of the basins of attraction of the studied methods, we also count number of points that convergent in order to make the observation clearer. This will be presented in the table bellow.

TABLE 1. Comparison of number of divergent points of iterative methods in solving $f(x)=0$ in complex plane

Function	Roots	AM36	FM14	NM9	ASM8
$f_1(x)$	$-0.5-0.866i$	208441	306731	322911	330759
	$-0.5+0.866i$	208495	306726	322911	330759
	1	333214	354458	354178	338482
	divergent	249850	32085	0	0
$f_2(x)$	-0.1	135786	151268	196936	135748
	0	630188	697464	603184	728504
	1	135786	151268	196936	135748
	divergent	98240	0	2944	0
$f_3(x)$	-3	10888	7836	12822	22562
	-1	449472	492164	487178	477438
	1	449063	492164	487178	477438
	3	10873	7836	12822	22562
	divergent	79704	0	0	0
$f_4(x)$	$-0.8090-0.5878i$	218324	202702	212490	219331
	$-0.809+0.5878i$	218324	202730	212490	219331
	$0.309-0.951i$	181127	190013	197329	193175
	$-0.309+0.951i$	181127	190000	197329	193175
	1.000	156328	180014	180362	174988
	divergent	44770	34541	0	0

In Table 1, we display the number of convergent points to each root of the tested function. Number of divergent points can be found in the row marked as "divergent". The basins of attraction of the methods for each tested function are given in Figure 1, Figure 2, Figure 3 and Figure 4, consecutively. Colors in the figures indicate the roots of the studied functions where black color marks the divergent area.

In the first example we have run all the methods to obtain the roots of $f_1(x) = 0$. The basins of attraction is given in Figure 1. The results show that NM9 and ASM8 perform the best, followed by FM14 and AM36. In this case, AM36 has approximately 25% number of divergent points where FM14 has 3% number of divergent points. Despite having the highest order of convergence among the discussed methods, AM36 performs the worst.

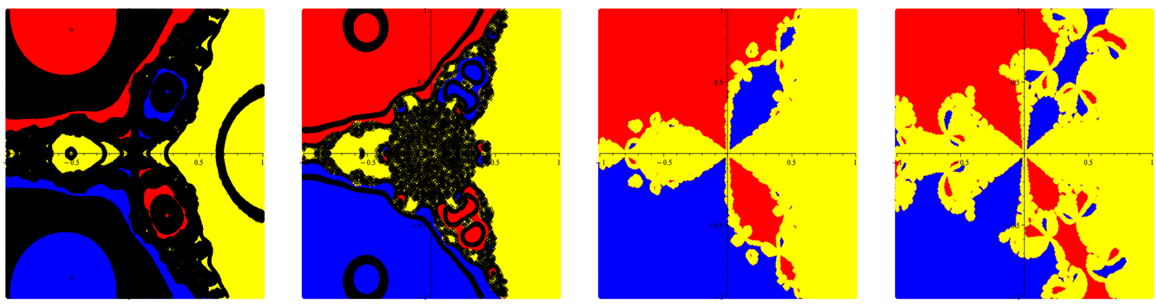
In the next experiment, we employed all the discussed method to get the roots of $f_2(x)$. The basins are given in Figure 2. In this part, FM14 and ASM8 succeed in sending all the initial points to converge followed by NM9 with just around 0.29% divergent points. However, AM36 has approximately 90.2% convergent points despite having the highest order of convergence of all.

A better results is evident in Table 1 where FM14, NM9 and ASM8 attain convergence for all of the four roots of $f_3(x)$. Nevertheless, AM36 are proven to be the weakest method. The basins of attraction are given in Figure 3.

As for $f_4(x)$, NM9 and ASM8 perform the best followed by FM14 as one can see in Figure 4. The basins of attraction of the latter method seem to have divergent points centered around the center of the plane and larger than of AM36. However, AM36 is the least favorable method since it has more divergent points than FM14.

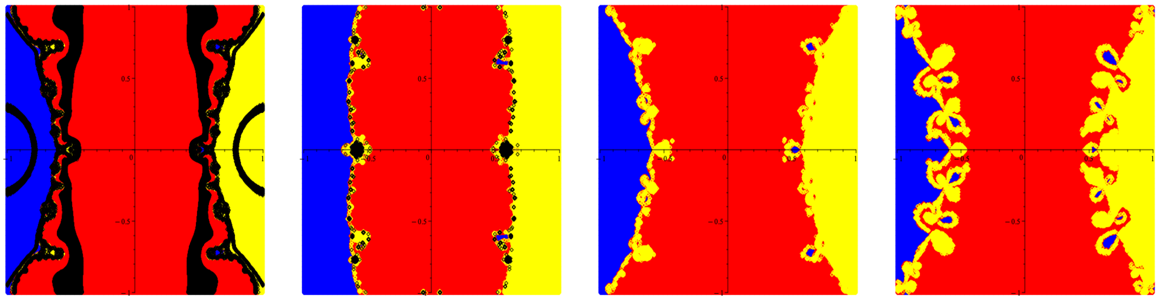
From this numerical simulations, it is evident that AM36 is the weakest method. Although it has the highest method, the efficiency index of this method is the lowest among all. The method also involves first and second derivative. which can cost more in the calculation. The second highest order method, FM14, has the highest efficiency index. On the contrary, the performance of the method has shown to be unfavorable as well. This method employs forward difference of order second and third to approximate first derivative. The third highest order method, NM9, outperforms the first two mentioned methods despite having the lowest efficiency index. The said method only consider first derivative and does not use any approximation to avoid the derivative. Ultimately, ASM8 performs the best regardless of its efficiency index being in between NM9 and FM14 and with the lowest order of convergence.

Our experiments show that order of convergence and efficiency index do not determine the performance of the methods. The derivatives involve in the scheme give less effect to the outcome of the methods as well. It is apparent that the choice of initial guesses play more important factor to show the success of an iterative method. In addition, our observations show that if we present a large number of initial guesses, there is a wider area of observations for the studied method.



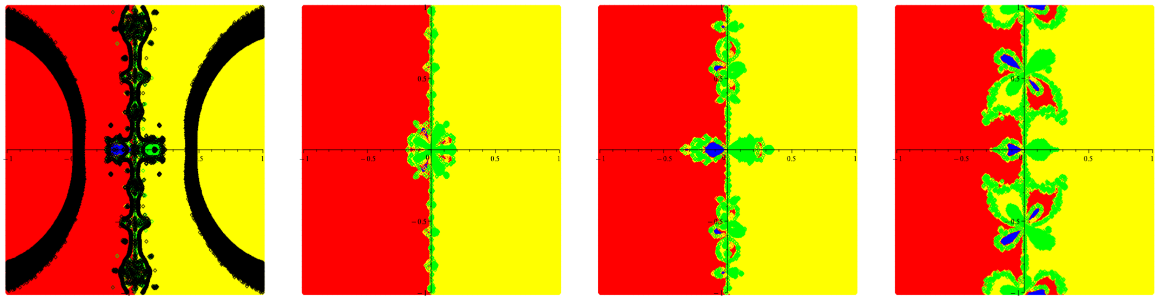
(A) AM36 (B) FM14 (C) NM9 (D) ASM8

FIGURE 1. Basins of attraction of iterative methods for $f(x) = x^3 - 1$



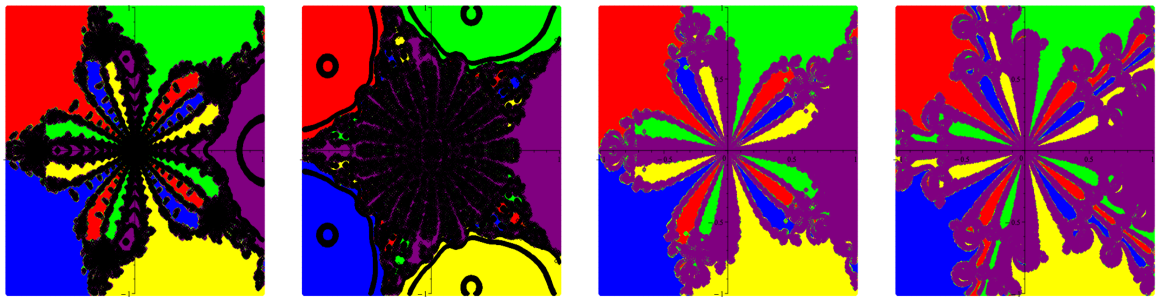
(A) AM36 (B) FM14 (C) NM9 (D) ASM8

FIGURE 2. Basins of attraction of iterative methods for $f(x) = x^3 - x$



(A) AM36 (B) FM14 (C) NM9 (D) ASM8

FIGURE 3. Basins of attraction of iterative methods for $f(x) = x^4 - 10x^2 + 9$



(A) AM36 (B) FM14 (C) NM9 (D) ASM8

FIGURE 4. Basins of attraction of iterative methods for $f(x) = x^5 - 1$

3. Conclusion

In this paper we have studied several root-finding methods with various order of convergence. We have compared number of convergent points attained by the methods and their basins of attraction.

We have concluded that the order of convergence of a method gives less effect to the performance of the method if we run the scheme on a large number of initial points. For future research, it is advisable to observe a new measure of efficiency of a root-finding method as it has been shown that order of convergence and efficiency index are not reliable in determining if a method is efficient if it is run with any initial guesses.

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