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Efficient Quadratic Programming Optimization via Staged Computational Procedures: Unconstrained Minimization and Constraint Exploration

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Abstract
In this paper, we present a computational framework for finding optimal solutions to quadratic programming problems. Our computational process is divided into three steps. Initially, we derive unconstrained minimization of the quadratic programming problem by solving simultaneous equations involving objective function derivatives and confirming its feasibility. Using this discovered point, we identify the violated constraints and direct our search to these specific constraints. The next stage defines the process for determining the unconstrained point on each active constraint violated by the objective function's optimal point. Moving on to the next stage, we use the constraint exploration technique to systematically seek the optimal constrained point at the intersections of two or more violated active constraints as candidates for the optimal solution. The feasibility of the unconstrained point is systematically checked at each level. If the unconstrained point is deemed feasible, then the optimal solution is obtained, and the optimal value of the objective function is found.

Keywords *: Quadratic, optimal, uncertainty, exploration, programming*

1. INTRODUCTION

Quadratic programming (QP) is a type of mathematics optimization problem that deals with the minimization or maximization of a quadratic objective function subject to linear equality and inequality constraints [1], [3]. The objective function has a quadratic shape, which is commonly represented by a positive semidefinite matrix, introducing curvature to the optimization landscape. QP is used in a variety of fields, including finance, engineering, and machine learning when the objective function has quadratic components. The purpose is to find the decision variable values that optimize the objective function while satisfying the stipulated linear constraints. The QP solutions can provide insights into efficient resource allocation, portfolio optimization, and other decision-making processes influenced by both linear and quadratic interactions between variables ([2], [5], [6]). Although it represents a natural transition from the theory of linear programming to nonlinear programming problems, there are some important differences between their optimal solution. If the optimum solution of a QP problem exists, then it is either an interior point or boundary point which is not necessarily an extreme point of the feasible region [4].

Uncertainty is widespread in Quadratic Programming Problems (QPP), particularly in portfolio optimization in the field of finance. The inherent unpredictability of asset returns, as well as the difficulty of precisely measuring quantities such as covariance and expected returns, contribute to this uncertainty. The covariance matrix, a fundamental component in QPP that represents asset risk, is especially susceptible to changes in historical data, market conditions, and unforeseen occurrences, adding to the uncertainty in optimization outcomes. The volatile nature of financial markets, as well

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as the impact of external factors such as economic shocks or geopolitical events, add to the unpredictability. Furthermore, transaction costs, another essential aspect, increase uncertainty because their estimation may not perfectly represent actual trading costs. Overall, addressing uncertainty in QPP entails using modeling methodologies that account for financial markets' intrinsic unpredictability, resulting in more resilient and adaptive portfolio optimization procedures ([7], [8], [9], [10] & [11]).

In the other hand, exploring violated constraints in Quadratic Programming Problems (QPP) entails looking into cases when the unconstrained optimal solution is obtained by solving simultaneous equations formed by equating all derivatives of the objective function to zero. Because, when solving a QPP, constraints are critical for ensuring that the generated solution corresponds to certain requirements set by the problem's environment. Violated constraints occur when the solution fails to match these requirements, which could be due to modeling flaws, data noise, or unforeseen complications in the optimization environment. Furthermore, the exploration of violated constraints gives useful information for model refinement. It aids in recognizing the limitations of the initial QPP formulation, leading changes to constraint definitions, relaxing constraints, or introducing more constraints to better depict the problem's real-world complexities. This iterative approach adds to improving the optimization model's accuracy and practical applicability, guaranteeing that the generated solutions fit more closely with the desired objectives and limitations in a variety of applications such as finance, engineering, and operations research.

Since the optimal solution of a QPP can be an interior or a boundary point, as well as the presence of uncertainty in the model, we would like to propose a method called exploration of violated constraints for exploring and obtaining the optimal solution of the original form of the QPP while avoiding the use of additional information. In addition, the concept of violated constraints is often an iterative process, involving a combination of mathematical analysis, algorithmic adjustments, and model refinement. Each iteration should bring you closer to a feasible and optimal solution to the considered problems.

The rest of the paper is organized as follows. In Section 2, fundamental notations and concepts are rigorously introduced, establishing a foundational framework for comprehending the intricacies of the problem and its associated violated constraints. The subsequent Section 3 formally formulates the QPP, delving into a thorough discussion on the exploration of violated constraints, encompassing both individual violations and the complex intersections of constraints. The culmination in Section 4 synthesizes these discussions, presenting a comprehensive algorithm meticulously designed to systematically determine the optimal solution for the QPP. This algorithm is distinguished by its noteworthy adaptability and efficiency in navigating the expansive solution space of the problem. Finally, Section 5 meticulously concludes the paper, succinctly summarizing the key findings and providing a road map for prospective avenues of research within this academic domain.

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2. PRELIMINARY

A quadratic function on $\mathbb{R}^n$ to be considered which is a positive semidefinite quadratic function defined by

$$f(x) = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} x_j d_{ij} x_i + \sum_{i=1}^{n} c_i x_i + q \tag{1}$$

where q, c_i and d_{ij} $(i,j = 1,...,n)$ are constant scalar quantities and can be written in compact form as

$$f(x) = \frac{1}{2}x^T Dx + C^T x + q \tag{2}$$

in which $D = (d_{ij})_{n\times n}$ is symmetric matrix and positive semidefinite, $c = (c_1,...,c_n)^T$, and $x = (x_1,...,x_n)^T$.

Without loss of generality [4], the linear function which represents a constraint is defined by

$$A^T x \ * b \tag{3}$$

where and A is an $m \times n$ matrix and b a vector in $\mathbb{R}^n$ and the symbol $\{*\}$ represents inequality or equality depending on the problem being considered.

The set of feasible region which the set of all solutions will be considered is closed set defined by

$$F = \{x | x \in \mathbb{R}^n, A^T x \leq b, x \geq 0\} \tag{4}$$

in which the symbol $\{*\}$ represents the $\leq$.

By letting $A = (a_1, ..., a_m)^T$ with

$$a_j = (a_{j1}, ..., a_{jm})^T \ (j = 1, ..., m), \tag{5}$$

the constraint inequalities can be written as

$$a_j^T x \leq b_j \ (j = 1, ..., m), \tag{6}$$

with the properties that the jth constraint is active if $a_j^T x = b_j$, is inactive if $a_j^T x < b_j$ and violated by x if $a_j^T x > b_j$.

3. PROBLEM FORMULATIONS

If we denoted $x^{(0)}$ as the unconstrained minimum point (minimizer) of (2) then we have

$$x^{(0)} = -D^{-1} c \tag{7}$$

As discussed previously, $x^{(0)}$ can be an interior or boundary point of a feasible region. However, there is a chance that $x^{(0)}$ is an exterior point. So, if $x^{(0)} \in F$, then $x(0)$ becomes the optimum solution to the problem under consideration.

Another advantage strict convexity features of $f(x)$ is that if $x^{(0)}$ is an exterior point, then the optimal solution, x^* of the considered problem must be on the boundary of the feasible region. As a result, x^* must be found on the active constraint or at the intersection of numerous active constraints, [4] and [11]. Through the discussion above, we can present the following results.

Corollary 3.1. *Suppose the quadratic programming problem is given by (2) subject to (4) then (i) If $x^{(0)}$ is determined by using (7) which is a unconstrained minimizer of $f(x)$ is deemed feasible then $x^{(0)}$ is referred to as the optimal solution of the problem. (ii)If $x(0) \notin F$, then $x*$ constrained minimum of $f(x)$ subject to F is on the boundary of the feasible region consisting of the equality of the violated constraints whose indexes in $V[x^{(0)}] = \{j | j \in \{i, ..., m\}, a_j^T x > b_j\}$ and denoted by set S.*

3.1. Searching the equality constraint point

Let us consider the QPP with $n = 3$. Suppose that the violated constraint is given by

$$ax + by + cz \leq d \tag{8}$$

and its equality constraint is

$$ax + by + cz = d. \tag{9}$$

By choosing three points in the equation (8), for instance $(d/a, 0, 0), (0, d/b, 0)$ and $(0, 0, d/c)$, and using α and β for the step size such that the point. Then, any point with direction α and β is given by

$$\left(\frac{d}{a}, 0, 0)\right) + \alpha\left(-\frac{d}{a}, \frac{d}{b}, 0\right) + \beta\left(-\frac{d}{a}, 0, \frac{d}{c}\right) \tag{10}$$

which lies on the plane (9 is uniquely determined since there is one to one correspondence between the point and its respected α and β. Therefore, by substituting this point into (3), the function of $f(\alpha, \beta)$ is obtained. The unconstrained minimum of $f(\alpha, \beta)$ can be achieved through minimizing $f(\alpha, \beta)$ with respect to α and β. Then we can obtain the point:

$$\left(\frac{d}{a} - \frac{d}{a}\alpha^* - \frac{d}{a}\beta^*, \frac{d}{b}\alpha^*, \frac{d}{c}\beta^*\right) \tag{11}$$

which denotes to unconstrained minimum of $f(x)$ on the equality constraint given by (8).

3.2. Searching on the intersection of active constraints

This subsection, will describe how to find the optimal point at the intersection of two active constraints. The point is then tested for feasibility. If it is feasible, it will be the ideal solution for the problem at hand. Otherwise, the intersection of three or more constraints must be checked.

Now consider two hyperplanes that are violated by the unconstrained minimum as follows:

$$a_i x + b_i y + c_i z = d_i,\ (i) \tag{12}$$

$$a_j x + b_j y + c_j z = d_j,\ (j) \tag{13}$$

which is sketched in following Figure 1.

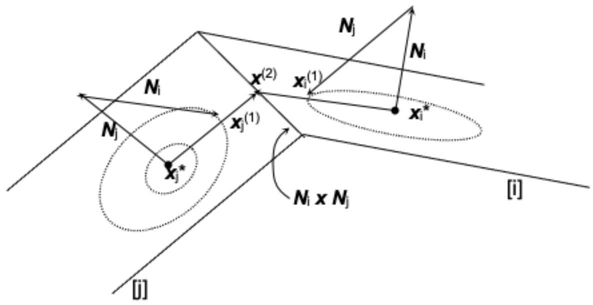

FIGURE 1. Intersection of two planes [i] and [j].

Figure 1, represents two constraints labeled by [i] and [j], respectively. The constrained minimum points on each constraint denoted by x_* and x_j* are determined in the previous section. The optimal point at the intersection of the two constraints can be calculated by using any point, x_i^1 in ith constraint together with the normal components of the both constraints described in the following result.

Given the contained minimum for ith and jth constraints, respectively.

$$x_i^* = (x_i, y_i, z_i) \tag{14}$$

$$x_j^* = (x_j, y_j, z_j) \tag{15}$$

The components of normal vector N for ith and jth constraints, respectively.

$$N_i = (a_i, b_i, c_i) \tag{16}$$

$$N_j = (a_j, b_j, c_j) \tag{17}$$

Theorem 3.2. *If the Corollary 3.1 is valid then*

(i): *The point $x_i^{(1)}$ on the ith constraint can be determined by*

$$x_i^{(1)} = x_i^* + N_i + \alpha N_j \tag{18}$$

in which α can be calculated by

$$\alpha = \frac{d_i + N_i \cdot N_i + x_i^* \cdot N_i}{N_i \cdot N_j} \tag{19}$$

(ii): *The unconstrained minimum point, $x^{(2)}$ on the intersection of ith and jth constraints can be determined by*

$$x^{(2)} = x_i^* + \beta(x_i^{(1)} - x_i^*) \tag{20}$$

where

$$\beta = \frac{d_j - x_j^* \cdot N_j}{N_i \cdot N_j + \alpha N_j \cdot N_i}$$

PROOF.

(i) The point $x_i^{(1)}$ must lie in the ith constraint iff

$$x_i^{(1)} \cdot N_i = d_i$$

Hence

$$\begin{aligned}(x_i^* + N_i + \alpha N_j) \cdot N_i &= d_i \\ x_i^* \cdot N_i + N_i \cdot N_i + \alpha N_j \cdot N_i &= d_i.\end{aligned}$$

Thus

$$\alpha = \frac{d_i + N_i \cdot N_i + x_i^* \cdot N_i}{N_i \cdot N_j}$$

(ii) Building on the assumption that point $x_i^{(1)}$ is within the ith constraint, additional investigation extends to the jth constraint. The condition states that point $x^{(2)}$ has to reside in the jth constraint if and only if it meets all of the jth constraint's prescribed conditions and restrictions. In essence, this means that the validity of $x^{(2)}$ is conditional on it meeting the specific criteria outlined by the jth constraint, emphasizing the importance of adherence to constraints in defining the feasibility and acceptability of points within the problem's solution space. So,

$$\begin{aligned} x^{(2)} \cdot N_j &= d_j \\ (x_i^* + \beta(x_i^{(1)} - x_i^*)) \cdot N_j &= d_j \\ x_i^* \cdot N_j + \beta(N_i + \alpha N_j) \cdot N_j &= d_j \end{aligned}$$

Thus, we obtain

$$\beta = \frac{d_j - x_j^* \cdot N_j}{N_i \cdot N_j + \alpha N_j \cdot N_i}.$$

4. THE OUTLINE OF ALGORITHM

The outcomes presented in the preceding section serve as a foundation for formulating an algorithm to attain the optimal solution for QPP. The outlined algorithm can be summarized as follows:

Step 1: Compute $x^{(0)}$, the unconstrained minimum of $f(x)$ by using (7).

Step 2: Start with a feasibility test to see if the initial answer $x^{(0)}$is feasible. If $x^{(0)}$ satisfies all of the constraints imposed by the problem, the algorithm terminates, and $x^{(0)}$ is regarded as the optimal solution to the QPP. If $x^{(0)}$ is outside the feasible region, indicating a constraint violation, identify and catalog the indices of the violated constraints. These indices are then added to the set $V[x^{(0)}]$ for further study and exploration. This procedure assures that the algorithm dynamically adjusts to the restrictions of the problem, iteratively improving its solution to optimality.

Step 3: Calculate x_j^*, which is the constrained minimum of the objective function $f(x)$ under the influence of equality constraint j, where j is a member of the set $V[x^{(0)}]$. If the resulting x_j^* for a certain $j \in V[^{(0)}]$ is regarded feasible, then x_j^* becomes the optimal solution for the QPP, causing the program to stop. If x_j^* is infeasible, extend the search to consider the possibility that the optimal solution exists at the intersection of two or more violated constraints identified by $x^{(0)}$. This investigation is being carried out in accordance with the approach outlined in [4].

This iterative procedure improves the algorithm's adaptability by systematically navigating the solution space to arrive at the best result.

5. CONCLUSION

In this paper, we address quadratic programming problems through the use of constraint exploration methods. The process of finding an optimal solution involves leveraging the concept of constraints being violated by the unconstrained. This innovative approach enables the efficient identification of optimal solutions to quadratic programming problems encountered in real-life scenarios. Through this methodology, complex problems in quadratic programming can be solved effectively and easily, contributing to practical applications in various domains.

REFERENCE

[1] Bernau, H. Quadratic programming problems and related linear complementarity problems. 1990, *J Optim Theory Appl*, **Volume** 65, pp. 209-222.

[2] J. Hall, A. Nurkanović, F. Messerer and M. Diehl, "A Sequential Convex Programming Approach to Solving Quadratic Programs and Optimal Control Problems With Linear Complementarity Constraints," in IEEE Control Systems Letters, **Volume** 6, pp. 536-541, 2022, doi: 10.1109/LCSYS.2021.3083467.

[3] I. M. Bomze, M. Dür, E. de Klerk, A. Quist, C. Roos, and T. Terlaky, 20 00, On copositive programming and standard quadratic optimization problems, *J. Global Optim.*, **Volume** 18, pp. 301-320.

[4] Ismail Bin Mohd and Yosza bin Dasril, 2000, Constraint exploration method for quadratic programming problem, *J. Applied Mathematics and Computation*. **Volume**: 112, pp. 161-170.

[5] Fernández-Navarro F, Martínez-Nieto L, Carbonero-Ruz M, Montero-Romero T. 2021, Mean Squared Variance Portfolio: A Mixed-Integer Linear Programming Formulation. *Mathematics*. **Volume** 9(3):223.

[6] Kassa, Semu Mitiku , & Tsegay, Teklay Hailay, 2018, An iterative method for trilevel quadratic fractional programming problems using fuzzy goal programming approach *Journal of Industrial Engineering International,* **Volume** 14(2), article id. , 264 pp.

[7] R. E. Bellman, L. A. Zadeh, 1970, Decision-Making in a Fuzzy Environment. *Management Science* **Volume** 17(4),pp. B-141-B-164.

[8] E. Ammar and H. A Khalifah,2003, Fuzzy portfolio optimization a quadratic programming approach, *J. of Chaos, Solitons and Fractals*. **Volume** 18, pp. 1045-1054.

[9] J. J Buckley, E. Esfandiar, and F Thomas, 2002, *Fuzzy mathematics in economics and engineering*, Physica-Verlag (2002)

[10] Petr Ya. Ekel, Marden Menezes, Fernando H. Schuffner Neto, 2007, Decision making in a fuzzy environment and its application to multicriteria power engineering problems, *Nonlinear Analysis: Hybrid Systems*, **Volume** 1(4), pp. 527-536,

[11] Y. Dasril, Z Zakaria and I.B Mohd, 2018, Using alpha-cuts and constraint exploration approach on Quadratic Programming Problem, *TELKOMNIKA*, **Volume** 16(6), pp. 2782-2790