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Hamiltonian and Hypohamiltonian of Generalized Petersen Graph ($GP_{n,6}$)

Putti Naura Adwita[1] and Sri Gemawati[1*]

[1]Department of Mathematics, University of Riau, Pekanbaru 28293, Indonesia
*Corresponding author's email: gemawati.sri@lecturer.unri.ac.id

Abstract
This article discusses the Hamiltonian and Hypohamiltonian properties of Generalized Petersen Graphs ($GP_{n,6}$ & $GP_{n,7}$). A Hamiltonian graph is a graph that has a Hamiltonian cycle; i.e. having a cycle that passes through each vertex exactly once. A Hypohamiltonian graph is if it is not a Hamiltonian graph, but if one vertex is removed it will be Hamiltonian. The Petersen graph is a cubic graph with ten vertices and fifteen edges and each vertex is of degree three. The generalized Petersen graph is denoted $GP_{n,k}$, for positive numbers n and k with $2 \leq 2k < n$. The Petersen graph is not a Hamiltonian graph, but is Hypohamiltonian. In the Generalized Petersen graph for $GP_{n,6}$ for $n \equiv 1(mod\ 13)$, $n \equiv 3(mod\ 13)$, $n \equiv 7(mod\ 13)$, $n \equiv 9(mod\ 13)$ is a Hamiltonian, for $n \equiv 0(mod\ 13)$ is a hypohamiltonian, and for $n \equiv 2(mod\ 13)$, $n \equiv 4(mod\ 13)$, $n \equiv 5(mod\ 13)$, $n \equiv 6(mod\ 13)$, $n \equiv 8(mod\ 13)$ neither.

Keywords: *Petersen graph, generalized Petersen graph, Hamiltonian, Hypohamiltonian*

1. INTRODUCTION

Mathematics is a basic science that is used as a thinking tool to solve problems in various fields of science. Mathematics has a very broad scope, one of which is graph theory. Graph theory is a unique field because its modeling applications are usefully for applications in various things such as transportation, communication networks, computer science, biology, economics, engineering, health and social sciences. Graphs are one of the applications used to date using theory. According to West et al. [13, h. 1], graph theory succeeded in solving its first problem in 1973, namely the problem of the Koningsberg bridge in the city of Koningsberg. In Russia there is the Pregal river which flows around the island of Kneiphof and branches into two tributaries. This problem was solved by a Swiss mathematician named Leonhard Euler. Euler's solution represents this problem in a graph with four landmasses as vertex and seven bridges as edges.

Until now, graph theory has developed in various fields of representation, with modeling applications that can be used to make it easier to analyze problems in graphs. In graph theory there are several properties of the connectedness of a graph that are very interesting to study, namely the Hamiltonian and Hypohamiltonian. A graph is called Hamiltonian if it has cycles that pass through all the vertices. The cycle of a graph that contains each vertex is called a Hamilton cycle. A graph is called Hypohamiltonian if every time one vertex is removed it becomes Hamiltonian. Based on the relationship between these two properties, it is very interesting to associate it with a generalized Petersen graph.

According to Potanka et al. [9, h. 32], the Petersen graph is known as a regular graph of 3-degree at all its vertices and has been generalized. The Petersen graph is very popular to study because it is unique, serves as an example of refutation in various places and has various interesting properties. In Ginting and Banjarnahor et al. [5] discussing the relationship between graph properties in Petersen graphs. Then, Wallis et al. [12, h. 34] discusses the properties of the Hamiltonian, Chen and Fan et al. [1] discuss the properties of the Hypohamiltonian. Furthermore, Ryjacek et al. [10] study the properties of the Hamiltonian in 3-connected independent graphs. The discussion which only focuses on independent graphs and the properties of the Hamiltonian only attracts the author's interest in discussing further the properties of the Hamiltonian and Hypohamiltonian in generalized Petersen graphs ($GP_{n,6}$).

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In the second part, several theoretical bases that support this research are explained. The third part discusses this problem, namely proving the existence of the Hamiltonian cycle and the validity of the Hamiltonian and Hypohamiltonian properties on generalized Petersen graphs ($GP_{n,6}$). Then continued to the fourth part by explaining the conclusions of the discussion of this article.

2. PRELIMINARIES

2.1 Graph

Graphs are a branch of mathematics that is widely used to describe various existing structures. According to Munir et al. [8, h, 356], the definition of a graph is as follows.

Definition 2.1 The graph G is a pair of sets ($V(G)$, $E(G)$) where $V(G)$ is a non-empty and finite set of objects called vertex, and $E(G)$ is a set of unordered pairs of different vertices in $V(G)$ are called edges. The set of vertices in G is denoted by $V(G)$ and the set of edges is denoted by $E(G)$. Whereas the number of elements in $V(G)$ is called the order of G and is denoted by $p(G)$ and the number of elements in $E(G)$ is called the measure and is denoted by $q(G)$.

Definition 2.2 Edge $e = (u, v)$ is called to connect vertex u and v if $e = (u, v)$ is an edge in the graph G, then u and v are called to be directly connected (adjacent), u and e and v and e are called to be directly related (incidents). Edge e is denoted by $e = uv$.

2.1.2 Simple Graph and Unsimple Graph

According to Kusmira and Taufiqurrochman et al. [6], graphs can be grouped based on the presence of rings or double edges in a graph, namely simple graphs and unsimple graphs.

Definition 2.3 A simple graph G is a graph that has ∀ vertices (V) ∄ edges (E), namely ring edges and double edges. In a simple graph, the edges are in unordered pairs. Meanwhile, a unsimple graph G is a graph that has ∀ vertices (V), ∃ circular edges or double edges or both. Below, Figure 1 (a) is an example of a simple graph and (b) is an example of a unsimple graph.

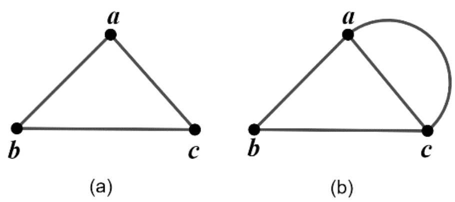

Figure 1. (a) Simple Graph, (b) Unsimple Graph

2.1.2 Directed Graph and Undirected Graph

Furthermore, Slamin et al. [11, h. 12] it is also explained that graphs can also be grouped into directed graphs and undirected graphs.

Definition 2.4 A directed graph G is a graph whose edges are not the same length and has a direction where (u, v) ≠ (v, u) and a sequence of pairs of vertices must be connected by different edges. Meanwhile, an undirected graph G is a graph whose edges are the same size where (u, v) = (v, u) and the order of pairs of vertices connected by the edges is not taken into account. An example can be seen in Figure 2 (a) is a directed graph and (b) is an undirected graph.

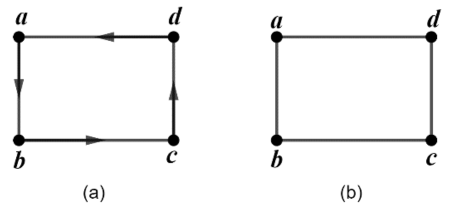

Figure 2. (a) Directed Graph, (b) Undirected Graph

2.2 Degree

According to West et al. [13, h. 34], the sum of the degrees of each vertex on a graph is called the graph degree. The degree of a vertex is the number of edges connected to that vertex.

Definition 2.5 The degree of vertices v in graph G, denoted by $d(v)$, is the number of edges in G that are invident to v. A vertex with degree zero is called an isolated vertex. The minimum degree of graph G is denoted by $\delta(G)$ and the maximum degree is denoted by $\Delta(G)$.

2.3 Connected Graph

In the connectivity of a graph, Makalew, Montolalu and Mananoas et al. [7] introduce several terms, namely walk, path, trail, and cycle.

Definition 2.6 A connected graph G is an undirected graph G if $\forall$ pairs of vertices u and v in the set $V \exists$ a path from u to v and if not, then G is called an unconnected graph.

Definition 2.7 A walk is a finite sequence of vertices and edges that begins and ends such that each edge is connected to the vertices before and after it. Let u and v be points on graph G. The walk $u - v$ on graph G is an alternating finite sequence. A walk that has no edges is called a trivial path**.**

Definition 2.8 A walk where all edges are different is called a trail. An open walk that passes through all different vertices is called a path. Therefore, every path is definitely a trail, but not all trails are paths. A closed walk with each edge distinct is called a circuit. A closed walk with different vertices at each vertex is called a cycle. Therefore, every cycle is definitely a circuit, but not all circuits are cycles.

2.4 Custom Graph

Following are several special graphs that have been introduced by Chia, Ong, and Arumugm et al. [2], Deo et al. [3, h. 2], Frick and Singleton et al. [4], Potanka et al. [9, h. 32], which are discussed, including regular graphs, complete graphs, cubic graphs, Petersen graphs, and generalized Petersen graphs.

Definition 2.9 A regular graph is a graph in which all vertices have the same degree. If each vertex has r-degree, then the graph is called a regular graph of r-degree. The complete graph K_n is a regular graph of (n-1)-degree. If graph G has n vertices and r-degree then graph G has $\frac{nr}{2}$ edges**.** In Figure 3 below is an example of a regular graph.

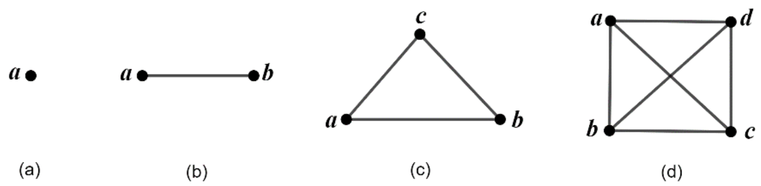

Figure 3. Regular Graph of Degree (a) 0, (b) 1, (c) 2, (d) 3

Definition 2.10 A complete graph is a graph that has $\forall$ vertices (V), $\exists$ edges (E) connected between every two vertices and is denoted by K_n, which is (n-1)-regular graph with order $p = n$ and size $q = \frac{n\{n-1)}{2}$**.** In Figure 3 above and Figure 4 below is an example of a complete graph

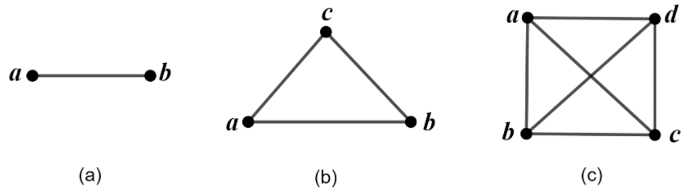

Figure 4. Complete Graph (a) K_1, (b) K_2, (c) K_3

Definition 2.11 A cubic graph is a graph where every vertex v has 3-degree, or is often called a regular graph of 3-degree. In Figure 5 below is an example of a cubic graph.

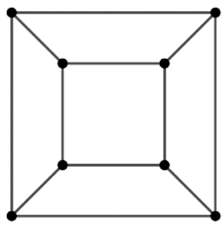

Figure 5. Cubic Graph

Definition 2.12 The Petersen graph is regular of 3-degree. In a Petersen graph all the vertices are of 3-degree so the Petersen graph is called a cubic graph with ten vertices and fifteen edges. Petersen graph is vertex-transitive and edge-transitive and is symmetrical. An example can be seen in Figure 6 is an example of Petersen graph.

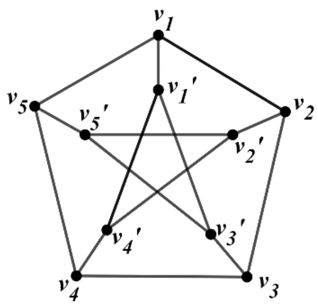

Figure 6. Petersen Graph

Definition 2.13 The Generalized Petersen graph is denoted by $GP_{n,k}$ where n and k are positive integers with $2 \leq 2k < n$, which is a graph with

$$V(GP_{n,k}) = \{u_0, u_1, \dots, u_{n-1}, \; v_0, v_1, \dots, v_{n-1}\},$$

$$E(GP_{n,k}) = \{u_i u_{\{i+1\}}, v_i v_{\{i+k\}}, u_i v_i \,|\, i = 0, 1, \dots, n-1\},$$

where the addition in the index $(i + 1)$, $(i + k)$ is modulo n. The Generalized Petersen graph $GP_{n,k}$ has three types of edges, namely outer edge, inner edge, and spoke. The outer edge connects vertices u_i and $u_{\{i+1\}}$. The inner edge connects vertices v_1 and $v_{\{i+k\}}$, while the spoke connects vertices u_i and v_1. The following in Figure 7 is an example of a generalized Petersen graph.

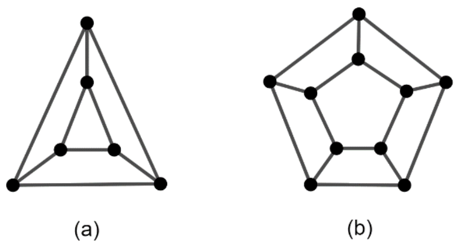

Figure 7. Generalized Petersen (a) $GP_{3,1}$, (b) $GP_{5,1}$

2.5 Properties of Hamiltonians and Hypohamiltonians

A graph is called Hamiltonian if it has a Hamilton cycle. A graph is called Hypohamiltonian if it is not Hamiltonian but can become Hamiltonian if one vertex is removed. Ginting and Banjarnahor et al. [5] the following definitions are explained.

Definition 2.14 A Hamiltonian graph is a graph that has a Hamiltonian cycle. A Hamiltonian path is a path ∀ point (*V*) traversed in graph *G* exactly only once, where the origin vertex $v_0 \neq$ end vertex v_n, while a closed Hamiltonian path is a path ∀ point (*V*) traversed in graph *G* exactly only once, where the origin vertex $v_0 \neq$ end vertex v_n. The following in Figure 8 is an example of Hamiltonian graph.

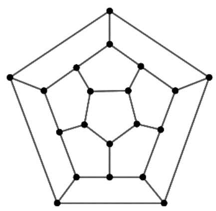

Figure 8. Hamiltonian Graph

Definition 2.15 A graph *G* is called Hypohamiltonian if the graph *G* is not Hamiltonian, but if one vertex (*v*) is deleted every time, then the subgraph $G - v$ is Hamiltonian. An example of a Hypohamiltonian graph can be seen in Figure 9. A graph *G* is said to be Hypohamiltonian if it satisfies the following definition.

(a) A graph *G* is called Hypohamiltonian if it is not Hamiltonian,
(b) If one vertex is removed from graph *G*, it will form a Hamiltonian cycle so that it is Hypohamiltonian.

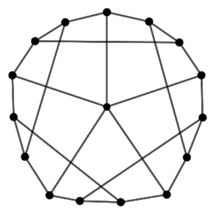

Figure 9. is an example of Hypohamiltonian Graph

3. RESULTS AND DISCUSSION

The author proves the existence of the Hamiltonian cycle and the validity of the Hamiltonian and Hypohamiltonian properties on the generalized Petersen graph ($GP_{n,6}$) for n = 13, 14, 15, 16, 17.18, 19, 20, 21, 22, 23, 24. The first step is to prove the validity of the Hamiltonian and Hypohamiltonian properties on the Petersen Graph. The Petersen graph can be represented in the following figure. First, it is proven that the Hamiltonian property applies to Petersen graphs which has been discussed in Imam et al. [14], with the results of the discussion showing that the Petersen graph is not a Hamiltonian because the Petersen graph does not have a Hamiltonian cycle. This will be proven by the following theorem:

Teorema 3.1 *The Petersen graph is not a Hamiltonian.*

BUKTI. It will be shown that the Petersen graph G is not Hamiltonian by using the edge-transitive Petesen graph property. Given the Petersen graph G in Figrure 10 as follows.

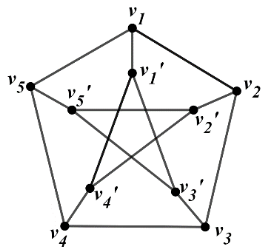

Figure 10. Petersen Graph ($GP_{5,2}$)

Based on Figure 10, a Petersen graph G is given. Suppose that

$$A = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\},$$

$$B = \{v_1v_1', v_2v_2', v_3v_3', v_4v_4', v_5v_5'\},$$

$$C = \{v_1'v_3', v_3'v_5', v_5'v_2', v_2'v_4', v_4'v_1'\},$$

is a subset of $E(G)$. Based on Definition 2.13, it is known that A is a set of outer edges, B is a set of spokes, and C is a set of inner edges. Based on Definition 2.12, suppose also that H is a cycle of the Petersen graph. Suppose H is a Hamiltonian. H must use an even number of sides of B, so that H has two or four sides because the maximum side that B has is five.

Based on Definition 2.12, the Petersen graph is edge transitive and is symmetrical, it can be assumed that $v_1v_1' \in E(H)$, so one of v_1v_2 or $v_5v_1 \in E(H)$ and it can also be assumed that $v_1v_2 \in E(H)$. Then, based on Definition 2.13, it is known that the Petersen graph is a cubic graph, so $v_5v_1 \notin E(H)$. Therefore, v_4v_5 and $v_5v_5' \in E(H)$. If H using two edges of B, namely that $v_1v_1'$and $v_5v_1 \in E(H)$, then $v_2v_3, v_3v_4 \in E(H)$. However, this situation requires that the vertices v_1' and v_5' have 3-degree on H. Consequently, $|E(H) \cap B| = 4$.

Based on the symmetry properties in Definition 2.12, there is one of v_2v_2', $v_4v_4' \in E(H)$. it can be assumed that $v_4v_4' \in E(H)$. Since $v_3v_4 \notin E(H)$, this requires v_2v_3, $v_3v_3' \in E(H)$. Then, there exists $v_2'v_4'$, $v_5'v_2', \in E(H)$due to $|E(H) \cap B| = 4$ and $v_2v_2' \notin E(H)$. This situation requires subcycles of v_2', v_5', v_5, v_4, v_4', v_2' $E(H)$. This is a contradiction and H not exist so the assumption that H is Hamiltonian is false. So, the Petersen graph is not a Hamiltonian. □

Next, it will be proven that the Hypohamiltonian property applies to graphs Petersen which has been discussed in Imam et al. [15], with the results of the discussion it was found that the Petersen graph is Hypohamiltonian because it meets the definition of Hypohamiltonian. This will be proven by the following theorem:

Teorema 3.2 *The Petersen graph is a Hypohamiltonian.*

BUKTI. It will be shown that the petersen graph G is Hypohamiltonian by using the definition of Hypohamiltonian. Based on Figure 10, it can be seen that the Petersen graph has two types of vertices, namely inner vertex and outer vertex. Based on Definition 2.15, the Petersen graph is said to be Hypohamiltonian if both conditions are met**.** Take any vertex on $GP_{5,2}$ and delete any vertex on the inner vertex. Assume that the vertex v_1' is deleted, so a Hamilton cycle can be created as follows.

$$H = \{v_1, v_2, v_2', v_4', v_4, v_3, v_3', v_5', v_5, v_1\}.$$

So, $GP_{5,2} - v_1'$ Hamiltonian. In $GP_{5,2}$ there is not exist the Hamilton cycle. However, each deletion of one vertex in $GP_{5,2}$ results in $GP_{5,2} - v_n$ having a Hamilton cycle. So, it is proven that $GP_{5,2}$ is Hamiltonian. □

Next, the author constructs steps to prove the validity of the Hamiltonian and Hypohamiltonian properties of the Petersen graph in general $GP_{n,6}$ with values n = 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24. The proof will be carried out by finding the existence of Hamilton cycles carried out one by one starting from the smallest n value to the largest n value. The author estimates that there will be three possibilities that apply to the generalized Petersen graph of $GP_{n,6}$, namely Hamiltonian, Hypohamiltonian, and neither. Based on Definition 2.13, for the $GP_{n,6}$, the values of n that satisfy $2 \leq 2k < n$ are more than or equal to 13. It will be shown that the Hamiltonian and Hypohamiltonian properties apply for each value of n. The author uses one of the mathematical software, namely Geogebra, to represent $GP_{n,6}$ in a figure.

(1) Generalized Petersen graph ($GP_{13,6}$)

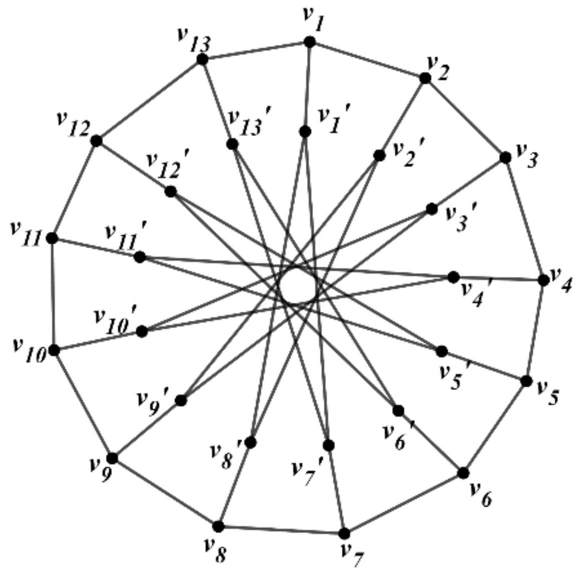

Figure 11. Generalized Petersen graph ($GP_{13,6}$)

Based on Figure 11, no Hamilton cycle was found in $GP_{13,6}$ so it is not Hamiltonian.

(2) Generalized Petersen graph ($GP_{14,6}$)

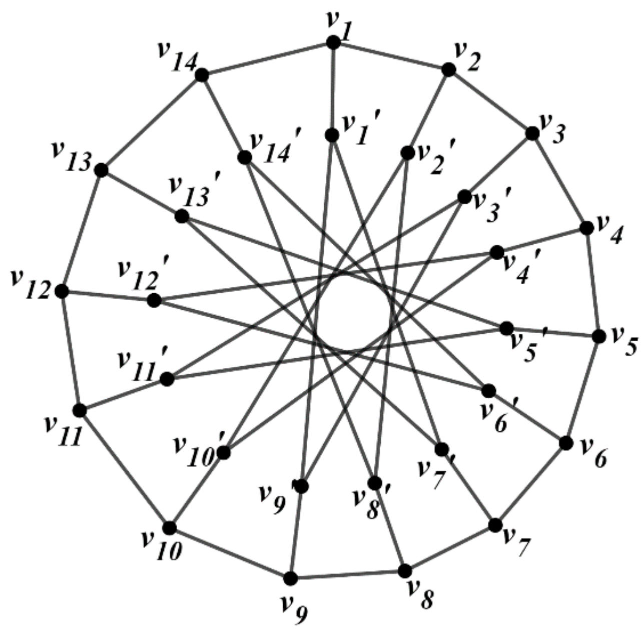

Figure 12. Generalized Petersen graph ($GP_{14,6}$)

Based on Figure 12, there is a Hamilton cycle as follows.

$$H = \{v_1, v_1', v_7', v_{13}', v_5', v_{11}', v_3', v_9', v_9, v_8, v_7, v_6, v_5, v_4, v_3, v_2, v_2', v_8', v_{14}', v_6', v_{12}', v_4', v_{10}', v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_1\}.$$

So, $GP_{14,6}$ is Hamiltonian.

(3) Generalized Petersen graph ($GP_{15,6}$)

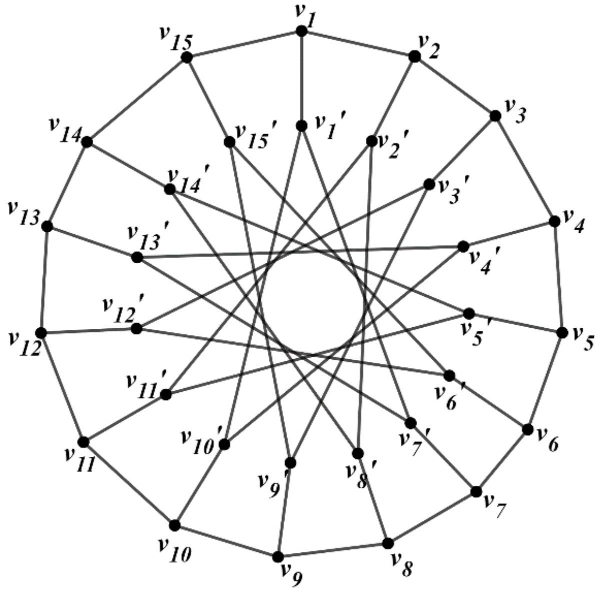

Figure 13. Generalized Petersen graph ($GP_{15,6}$)

Based on Figure 13, no Hamilton cycle was found in $GP_{15,6}$ so it is not Hamiltonian.

(4) Generalized Petersen graph ($GP_{16,6}$)

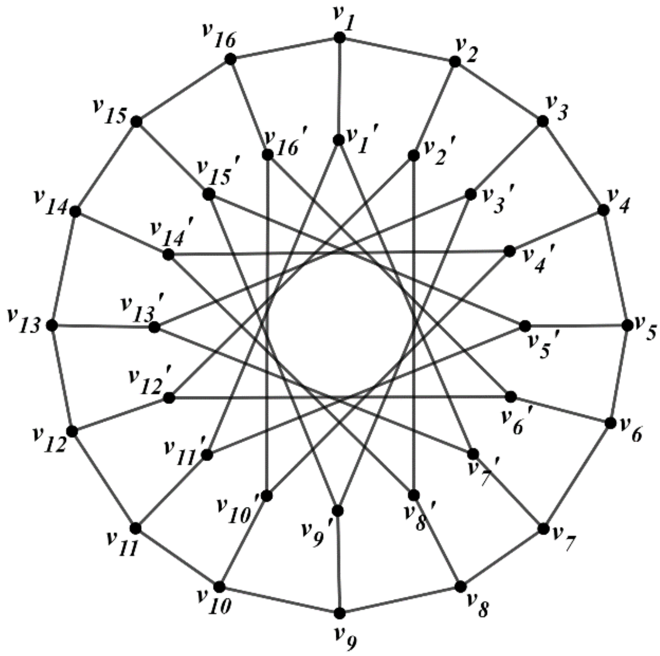

Figure 14. Generalized Petersen graph ($GP_{16,6}$)

Based on Figure 14, there is a Hamilton cycle as follows.

$$H = \{v_1, v_1', v_7', v_{13}', v_3', v_9', v_{15}', v_5', v_{11}{}', v_{11}, v_{10}, v_9, v_8, v_7, v_6, v_5, v_4, v_3, v_2\ v_2', v_8', v_{14}', v_4', v_{10}', v_{16}', v_6', v_{12}', v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_1\}.$$

So, $GP_{16,6}$ is Hamiltonian.

(5) Generalized Petersen graph ($GP_{17,6}$)

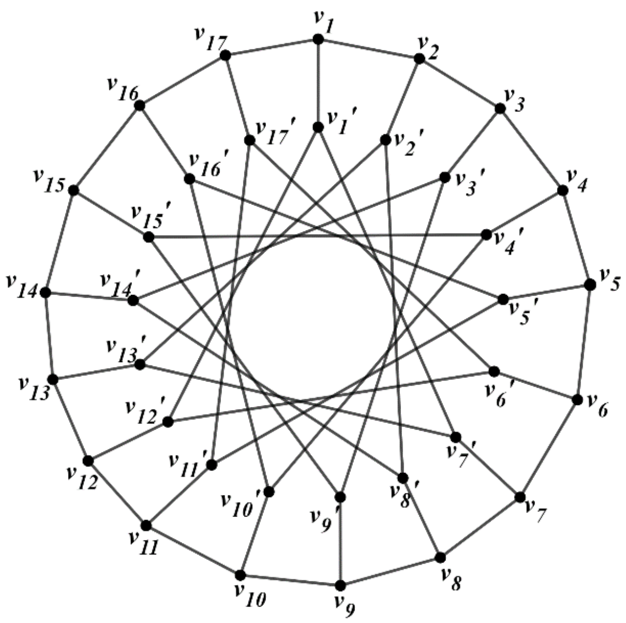

Figure 15. Generalized Petersen graph ($GP_{17,6}$)

Based on Figure 15, no Hamilton cycle was found in $GP_{17,6}$ so it is not Hamiltonian.

(6) Generalized Petersen graph ($GP_{18,6}$)

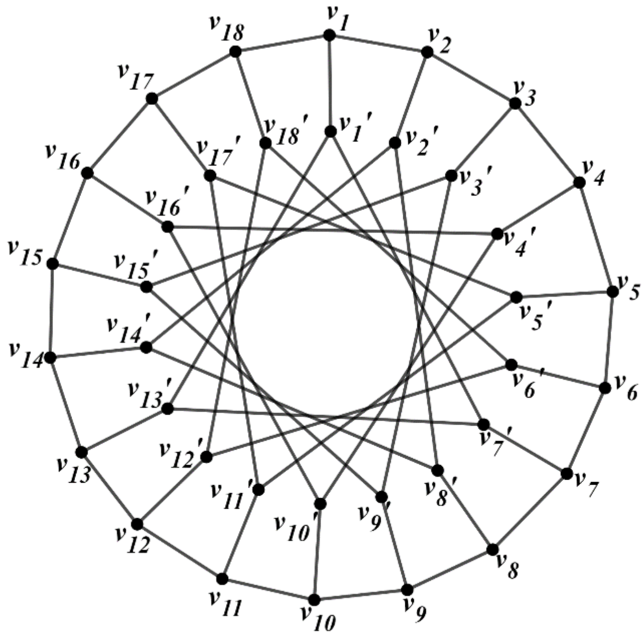

Figure 16. Generalized Petersen graph ($GP_{18,6}$)

Based on Figure 16, no Hamilton cycle was found in $GP_{18,6}$ so it is not Hamiltonian.

(7) Generalized Petersen graph ($GP_{20,6}$)

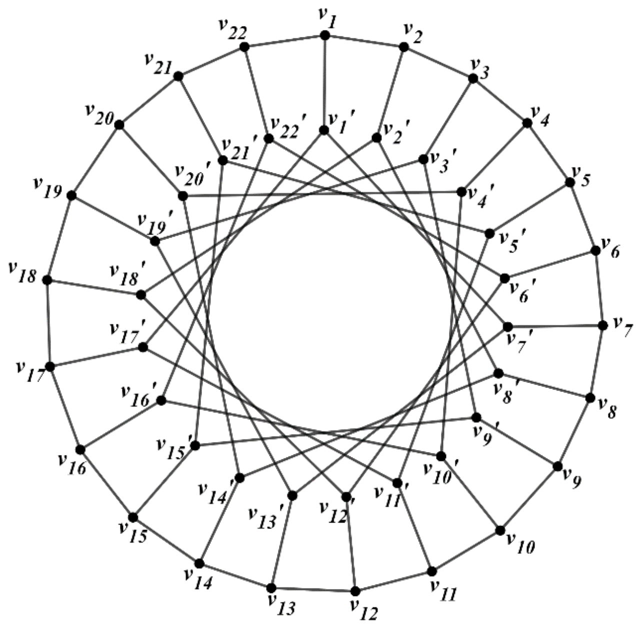

Figure 17. Generalized Petersen graph ($GP_{20,6}$)

Based on Figure 17, there is a Hamilton cycle as follows.

$$H = \{v_1, v_1', v_7', v_{13}', v_{19}', v_5', v_{11}', v_{17}', v_3', v_9', v_{15}', v_{15}, v_{14}, v_{13}, v_{12}, v_{11}, v_{10}, v_9, v_8, v_7, v_6, v_5, v_4, v_3, v_2, v_2', v_8', v_{14}', v_{20}', v_6', v_{12}', v_{18}', v_4', v_{10}', v_{16}', v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_1\}.$$

So, $GP_{20,6}$ is Hamiltonian.

(8) Generalized Petersen graph ($GP_{21,6}$)

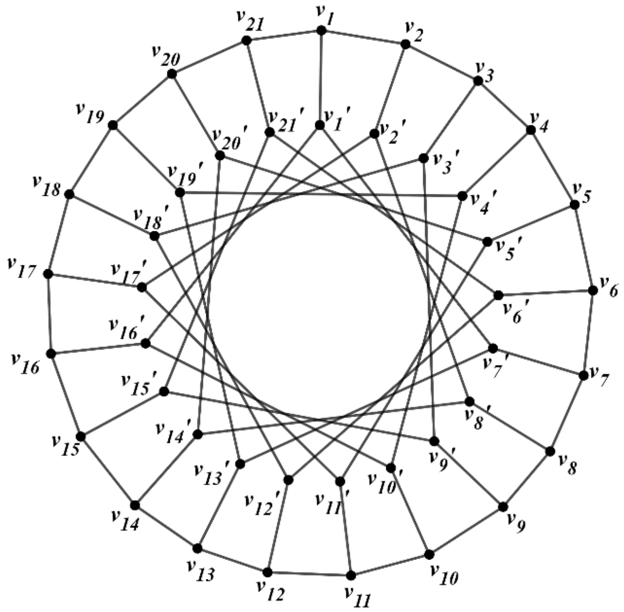

Figure 18. Generalized Petersen graph ($GP_{21,6}$)

Based on Figure 18, no Hamilton cycle was found in $GP_{21,6}$ so it is not Hamiltonian.

(9) Generalized Petersen graph ($GP_{22,6}$)

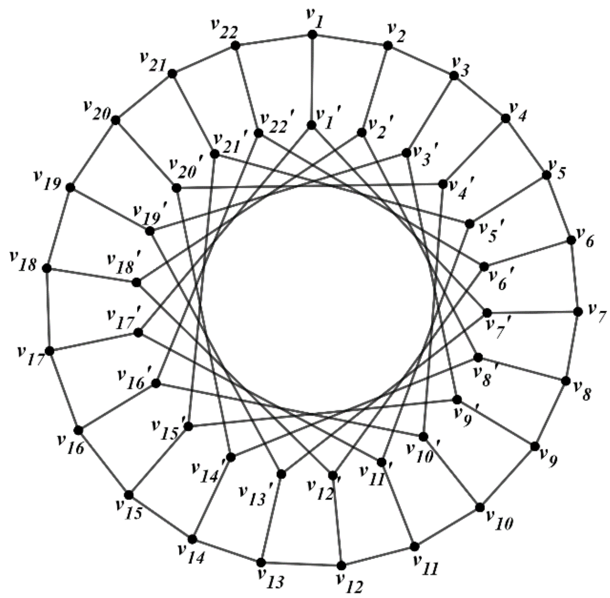

Figure 19. Generalized Petersen graph ($GP_{22,6}$)

Based on Figure 19, there is a Hamilton cycle as follows.

$$H = \{v_1, v_1', v_7', v_{13}', v_{19}', v_3', v_9', v_{15}', v_{21}', v_5', v_{11}', v_{17}', v_{17}, v_{16}, v_{15}, v_{14}, v_{13}, v_{12},$$

$v_{11}\ v_{10}, v_9, v_{8,} v_7, v_6, v_5, v_4, v_3, v_{2,} v'_2, v'_8, v'_{14}, v'_{20}, v'_4, v'_{10}, v'_{16}, v'_{22}, v'_6, v'_{12},$
$v'_{18}, v_{18}, v_{19,} v_{20}, v_{21}, v_{22}, v_1\}$.

So, $GP_{22,6}$ is Hamiltonian.

(10) Generalized Petersen graph ($GP_{23,6}$)

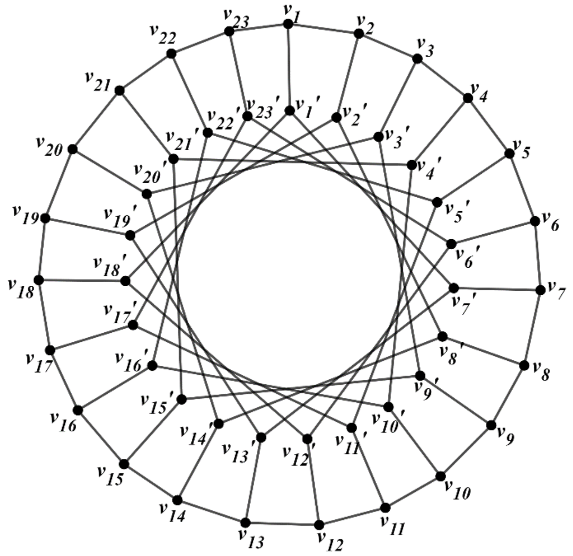

Figure 20. Generalized Petersen graph ($GP_{23,6}$)

Based on Figure 20, no Hamilton cycle was found in $GP_{23,6}$ so it is not Hamiltonian.

(11) Generalized Petersen graph ($GP_{24,6}$)

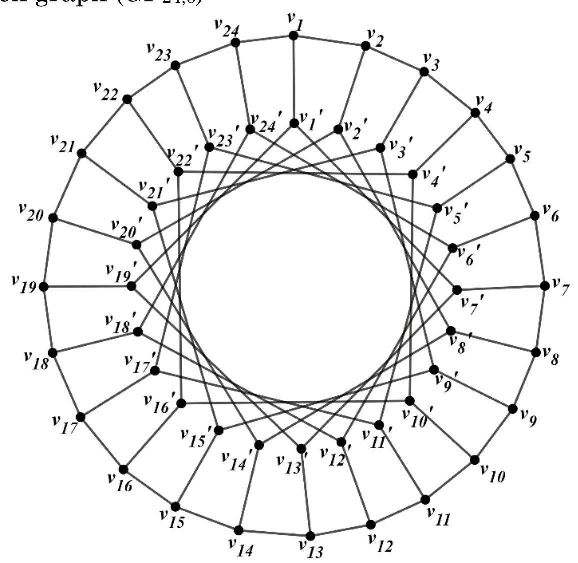

Figure 21. Generalized Petersen graph ($GP_{24,6}$)

Based on Figure 21, no Hamilton cycle was found in $GP_{24,6}$ so it is not Hamiltonian.

Based on the discussion above, it is found that in $GP_{n,6}$ with $n \equiv 1(mod\ 13)$, $n \equiv 3(mod\ 13)$, $n \equiv 7(mod\ 13)$, $n \equiv 9(mod\ 13)$ we find the existence of a Hamiltonian cycle so that it is a

Hamiltonian. Whereas in $GP_{n,6}$ with $n \equiv 0(mod\ 13)$, $n \equiv 2(mod\ 13)$, $n \equiv 4(mod\ 13)$, $n \equiv 5(mod 13)$, $n \equiv 6(mod13)$, $n \equiv 8(mod\ 13)$, $n \equiv 10(mod13)$, $n \equiv 11(mod\ 13)$ no Hamiltonian cycle was found so there are two possibilities, namely Hypohamiltonian or neither.

Teorema 3.3 *Generalized Petersen graph $GP_{n,6}$ with $n \equiv 1(mod\ 13)$, $n \equiv 3(mod\ 13)$, $n \equiv 7(mod\ 13)$, $n \equiv 9(mod\ 13)$ are Hamiltonian.*

BUKTI. It will be proven that the Petersen graph is generalized $GP_{n,6}$ with $n \equiv 1(mod\ 13)$, $n \equiv 3(mod\ 13)$, $n \equiv 7(mod\ 13)$, $n \equiv 9(mod\ 13)$ are Hamiltonians by indicating the existence of the Hamilton cycle. In the generalized Petersen graph $GP_{n,6}$ with $n \equiv 1(mod\ 13)$, $n \equiv 3(mod\ 13)$, $n \equiv 7(mod\ 13)$, $n \equiv 9(mod\ 13)$ there is a cycle that passes through all vertices exactly once starting at the starting vertex namely v1 and returns to the end point, namely v1. The cycle pattern is as follows.

$$H = \{v_1, v_1', v_7', v_{13}', \dots, v_{n-5}', v_{n-5}, \dots, v_2, v_2', \dots,\ v_n', \dots, v_{n-4}', v_{n-4}, v_n, , v_1\}.$$

The cycle obtained is a Hamilton cycle. So, it is proven $GP_{n,6}$ with $n \equiv 1(mod\ 13)$, $n \equiv 3(mod\ 13)$, $n \equiv 7(mod\ 13)$, $n \equiv 9(mod\ 13)$ is a Hamiltonian. □

Teorema 3.4 *Generalized Petersen graph $GP_{n,6}$ with $n \equiv 0(mod\ 13)$, $n \equiv 2(mod\ 13)$, $n \equiv 4(mod\ 13)$, $n \equiv 5(mod\ 13)$, $n \equiv 6(mod\ 13)$, $n \equiv 8(mod\ 13)$, $n \equiv 10(mod13)$, $n \equiv 11(mod\ 13)$ are not Hamiltonian.*

BUKTI. It will be proven that the Petersen graph is $GP_{n,6}$ with $n \equiv 0(mod\ 13)$, $n \equiv 2(mod\ 13)$, $n \equiv 4(mod\ 13)$, $n \equiv 5(mod\ 13)$, $n \equiv 6(mod\ 13)$, $n \equiv 8(mod\ 13)$, $n \equiv 10(mod13)$, $n \equiv 11(mod\ 13)$ are not Hamiltonians by showing the absence of Hamiltonian cycles.

In $GP_{n,6}$ with $n \equiv 0(mod\ 13)$, there is a cycle as follows.

$$H = \{v_1, v_1', v_7', v_{13}', v_6', v_{12}', v_5', v_{11}', v_4{}', v_{10}{}', v_3{}', v_9{}', v_2{}', v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_1\}.$$

It can be seen that vertex $v_8{}'$ is not crossed. So $GP_{n,6}$ with $n \equiv 0(mod\ 13)$ is not Hamiltonian.

In $GP_{n,6}$ with $n \equiv 2(mod\ 13)$, there is a cycle as follows.

$$H = \{v_1, v_1', v_7', v_{13}', v_4', v_{10}', v_{10}, v_9, v_8, v_7, v_6, v_5, v_4, v_3, v_2, v_2', v_8', v_{14}', v_5', v_{11}', v_{11}, v_{12}, v_{12}', v_3', v_9', v_{15}', v_6', v_{12}', v_{12}, v_{13}, v_{14}, v_{15}, v_1\}.$$

It can be seen that vertex v_{12} dan $v_{12}{}'$ are skipped twice. So $GP_{n,6}$ with $n \equiv 2(mod\ 13)$ is not Hamiltonian.

In $GP_{n,6}$ with $n \equiv 4(mod\ 13)$, there is a cycle as follows.

$$H = \{v_1, v_1', v_7', v_{13}', v_2', v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_8', v_{14}', v_3', v_9', v_{15}' v_4', v_{10}', v_{16}', v_5', v_{11}', v_{17}', v_6', v_{12}', v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_1\}.$$

It can be seen that vertex v_9, v_{10}, v_{11} are not crossed. So $GP_{n,6}$ with $n \equiv 4(mod\ 13)$ is not Hamiltonian.

In $GP_{n,6}$ with $n \equiv 5(mod\ 13)$, there is a cycle as follows.

$$H = \{v_1, v_1', v_7', v_{13}', v_{13}, v_{12}, v_{11}, v_{10}, v_9, v_8, v_7, v_6, v_5, v_4, v_3, v_2, v_2', v_8', v_{14}', v_{14}, v_{15}, v_{15}', v_3', v_9', v_{15}', v_{15}, v_{16}, v_{16}', v_4', v_{10}', v_{16}', v_{16}, v_{17}, v_{17}', v_5', v_{11}', v_{17}', v_{17}, v_{18}, v_{18}', v_6', v_{12}', v_{18}', v_{18}, v_1\}.$$

It can be seen that vertex $v_{15}, v_{15}', v_{16}, v_{16}', v_{17}, v_{17}', v_{18}, v_{18}'$ are skipped twice. So $GP_{n,6}$ with $n \equiv 5(mod\ 13)$ is not Hamiltonian.

In $GP_{n,6}$ with $n \equiv 6(mod\ 13)$, there is a cycle as follows.

$$H = \{v_1, v_1', v_7', v_{13}', v_{19}', v_6{}', v_{12}', v_{18}', v_5', v_{11}', v_{17}', v_4{}', v_{10}', v_{16}', v_3', v_9', v_{15}'\\ v_2', v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_8', v_{14}', v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_1\}.$$

It can be seen that vertex $v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}$ are not crossed. So $GP_{n,6}$ with $n \equiv 6(mod\ 13)$ is not Hamiltonian.

In $GP_{n,6}$ with $n \equiv 8(mod\ 13)$, there is a cycle as follows.

$$H = \{v_1, v_1', v_7', v_{13}', v_{19}', v_4', v_{10}', v_{16}', v_{16}, v_{15}, v_{14}, v_{13}, v_{12}, v_{11}, v_{10}, v_9,\\ v_8, v_7, v_6, v_5, v_4, v_3, v_2, v_2', v_8', v_{14}', v_{20}', v_5', v_{11}', v_{17}', v_{17}, v_{18}, v_{18}',\\ v_3', v_9', v_{15}', v_{21}', v_6', v_{12}', v_{18}', v_{18}, v_{19}, v_{20}, v_{21}, v_1\}.$$

It can be seen that vertex v_{18} and $v_{18}{}'$ are skipped twice. So $GP_{n,6}$ with $n \equiv 8(mod\ 13)$ is not Hamiltonian.

In $GP_{n,6}$ with $n \equiv 10(mod\ 13)$, there is a cycle as follows.

$$H = \{v_1, v_1', v_7', v_{13}', v_{19}', v_2', v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_8', v_{14}', v_{20}', v_3', v_9', v_9, v_{10},\\ v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{15}', v_{21}', v_4', v_{10}', v_{16}', v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22},\\ v_{22}', v_5', v_{11}', v_{17}', v_{23}', v_{23}, v_1\}.$$

It can be seen that vertex v_6, v_{12}', v_{18}' are not crossed. So $GP_{n,6}$ with $n \equiv 10(mod\ 13)$ is not Hamiltonian.

In $GP_{n,6}$ with $n \equiv 11(mod\ 13)$, there is a cycle as follows.

$$H = \{v_1, v_1', v_7', v_{13}', v_{19}', v_4', v_{10}', v_{16}', v_{16}, v_{15}, v_{14}, v_{13}, v_{12}, v_{11}, v_{10}, v_9,\\ v_8, v_7, v_6, v_5, v_4, v_3, v_2, v_2', v_8', v_{14}', v_{20}', v_5', v_{11}', v_{17}', v_{17}, v_{18}, v_{18}',\\ v_3', v_9', v_{15}', v_{21}', v_6', v_{12}', v_{18}', v_{18}, v_{19}, v_{20}, v_{21}, v_1\}$$

It can be seen that vertex $v_{21}, v_{21}', v_{22}, v_{22}', v_{23}, v_{23}', v_{24}, v_{24}'$ are skipped twice. So $GP_{n,6}$ with $n \equiv 11(mod\ 13)$ is not Hamiltonian.

So, it is proven that $GP_{n,6}$ with $n \equiv 0(mod\ 13)$, $n \equiv 2(mod\ 13)$, $n \equiv 4(mod\ 13)$, $n \equiv 5(mod\ 13)$, $n \equiv 6(mod\ 13)$, $n \equiv 8(mod\ 13)$ are not Hamiltonian. □

Based on the discussion in the previous section, it has been proven that there are some $GP_{n,6}$ which are not Hamiltonian, so there are two possibilities, namely that the Hypohamiltonian property applies or neither. Furthermore, in this section further proof is made of the validity of Hypohamiltonian properties.

Teorema 3.7 *Generalized Petersen graph $GP_{n,6}$ with $n \equiv 0(mod\ 13)$ is Hypohamiltonian.*

BUKTI. It will be shown that the Petersen graph is generalized $GP_{n,6}$ with $n \equiv 0(mod\ 13)$ is Hypohamiltonian. Based on Definition 2.15, the generalized Petersen graph $GP_{n,6}$ with $n \equiv 0(mod\ 13)$ already satisfies the first condition, namely it is not a Hamiltonian. Next, check the applicability of the second condition. Take any point on the inner vertex. Suppose that point $v_1{}'$ is deleted, so that it will form a cycle starting from the initial point v_1 and returning to point v_1. The cycle is as follows.

$$H = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_7', v_{13}', v_6', v_{12}', v_5', v_{11}', v_4', v_{10}', v_3',\\ v_9', v_2', v_8', v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_1\}$$

The cycle obtained is a Hamiltonian cycle so $GP_{n,6} - v_1{}'$ Hamiltonian. Both conditions are met, so it can be concluded that $GP_{n,6}$ with $n \equiv 0(mod\ 13)$ is Hypohamiltonian. □

4. CONCLUSIONS

Based on the discussion above, the properties of the Hamiltonian and Hypohamiltonian on Petersen graphs and Petersen graphs generalized $GP_{n,6}$ and $GP_{n,7}$ can be concluded that The Petersen graph is not Hamiltonian, but Hypohamiltonian. The generalized Petersen graph $GP_{n,6}$ for $n \equiv 1(mod\ 13)$, $n \equiv 3(mod\ 13)$, $n \equiv 7(mod\ 13)$, $n \equiv 9(mod\ 13)$ are Hamiltonian. For $n \equiv 0(mod\ 13)$ it is Hypohamiltonian. Meanwhile, $n \equiv 2(mod\ 13)$, $n \equiv 4(mod\ 13)$, $n \equiv 5(mod\ 13)$, $n \equiv 6(mod\ 13)$, $n \equiv 8(mod\ 13)$ are neither. The generalized Petersen graph $GP_{n,7}$ for $n \equiv 1(mod\ 15)$ and $n \equiv 9(mod\ 15)$ are Hamiltonian. For $n \equiv 0(mod\ 15)$, $n \equiv 2(mod\ 15)$, and $n \equiv 8(mod\ 15)$ are Hypohamiltonian. For $n \equiv 3(mod\ 15)$, $n \equiv 4(mod\ 15)$, $n \equiv 5(mod\ 15)$, $n \equiv 6(mod\ 15)$, $n \equiv 7(mod\ 15)$ neither.

REFERENCE

1. Chen, F., and Fan, G., Fulkerson-covers of hypohamiltonian graphs, 2015, *discrete applied mathematics*, **Volume:**186 pp 66-73.
2. Chia, G., L., Ong, S., H., and Arumugam, S., *Hamilton cycles in cubic graphs, 2007,* International Journal of Graphs and Combinatorics, **Volume:**3 pp 251-259.
3. Deo, N., 1989., *Graph Theory with Applications to Engineering and Computer Science*, Dover Publication, inc, New York.
4. Frick, M., and Singleton, J., Cubic Maximal Nontraceable Graphs, 2007, *Discrete Mathematics*, **Volume:** 307 pp 885-891.
5. Ginting, J., and Banjarnahor, H., Petersen graphs with some related properties in graph theory, 2016, *Karismatika*, **Volume:**2.
6. Kusmira, M., and Taufiqurrochman, *Utilization of graph applications in making the 05 Tasikmalaya public transportation route,* 2017.
7. Makalew, R., A., M., Montolalu, C., E., J., C., and Mananoas, M., L., Hamiltonian path in 4-connected graphs, 2020, *Journal of Mathematics and Applications*, **Volume:**9 pp 181-188.
8. Munir, R., 2010., *Discrete Mathematics 4th Edition*, Informatika, Bandung.
9. Potanka, K., S.,1998., Groups, Graphs, and Symmetry-Breaking, *M.Sc Thesis, Faculty of the*
10. *Virginia Polytechnic Institute and State University*.
11. Ryjacek, Z., Vrana, P., and Xiong, L., Hamiltonian Properties of 3-Connected Caw, Hourglass-Free Graphs, 2018, *Discrete Mathematics*, **Volume:** 341 pp1806-1815.
12. Slamin, 2023., *Graph Theory and Its Applications Second Edition,* R12 Group, Surabaya.
13. Wallis, W., D., 2007., *A Beginner's Guide to Graph Theory Second Edition,* Bikhauser, Boston.
14. West, D., B., 2001., *Introduction to Graph Theory Second Edition,* Universitas Illinois, Urban.
15. Imam, D., 2012., Sifat Hamiltonian dan Hipohamiltonian pada Graf Petersen Diperumum ($GP_{n,1}$ & $GP_{n,2}$*), Skripsi, Universitas Islam Maulana Malik Ibrahim*.