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Hamilton Cycle on the Wheel Graph

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Abstract

This article discusses the existence of the Hamilton cycle in the wheel graph by constructing steps to find the existence of the Hamilton cycle. A graph that has a Hamilton cycle is called a Hamilton graph, A circle graph is a graph where each vertex has a degree of two, denoted by $\boldsymbol{C_n}$. A graph obtained by adding a central vertex to a circle graph and connecting it to all the vertices of the circle graph is called a wheel graph, denoted by $\boldsymbol{W_n}$. If the wheel graph $\boldsymbol{W_n}$ has $\boldsymbol{m}$ where $\boldsymbol{m}$ is the number of $\boldsymbol{W_n}$ that replaces each point in $\boldsymbol{W_n}$ then it can be denoted by $\boldsymbol{W_n^m}$. Then, $\boldsymbol{n}$ in the wheel graph $\boldsymbol{W_n^m}$ is the number of outermost points of $\boldsymbol{W_n^m}$ added to $\boldsymbol{1}$ point located in the center. Based on the construction, it is found that there is a Hamilton cycle in the wheel graph. In the wheel graph $\boldsymbol{W_n}$ contains Hamilton cycle for $\boldsymbol{n \geq 3}$. Furthermore, the wheel graph $\boldsymbol{W_n^m}$ also contains Hamilton cycle for $\boldsymbol{n \geq 3}$ and $\boldsymbol{m \geq 1}$, but the image of the wheel graph $\boldsymbol{W_n^m}$ is only perfectly drawn for $\boldsymbol{n = 2k}$ where $\boldsymbol{k}$ is an integer. This is because there are colliding edges in the wheel graph for $\boldsymbol{n = 2k - 1}$ where $\boldsymbol{k}$ is an integer.

Keywords: *Wheel graph, Circle graph, Hamilton graph, Hamilton cycle*

1. INTRODUCTION

Over time, mathematics has developed until it has become a powerful thinking tool and is often used by scientists to solve complex problems. One branch of mathematics is Graph Theory. Graph theory is a field that models relationships between objects through graph structures to be applied in various things such as transportation, computer science, biology, economics, engineering, informatics, linguistics, health and social sciences. Graphs are one of the applications used to date using theory. According to [4, h. 3], Graph theory was born in 1736 through Euler's writings which contained efforts to solve the problem of the Konigsberg bridge which was very famous in Europe. The problem with the bridge is whether it is possible to cross each of the seven bridges in the city of Königsberg exactly once and return to the original place. This problem was successfully solved by Euler with a simple proof modeled in graph form. Euler's solution represents this problem in a graph with four landmasses as vertices and seven bridges as edges.

Currently, graph theory has developed in various fields of representation with modeling applications that can be used to make it easier to analyze problems in graphs. In graph theory there are several properties of the connectedness of a graph that are very interesting to study, namely Hamilton. A graph is said to be Hamiltonian if it has a cycle that passes through all the points. The cycle of a graph that contains each vertex with the same starting point as the ending point is called a Hamilton cycle. Apart from that, in graph theory there are also several special graphs in simple graphs, namely graphs that do not have double sides or rings, one of which is circle graphs and wheel graphs. A circle graph is a graph where each vertex is of degree two, denoted by C_n. A graph obtained by adding a center point to a circle graph and connecting that center point to all the points on the circle graph is called a wheel graph.

In [5] we discussed the Hamilton cycle in cubic graphs. In addition, [8] presents a discussion of vertices and minimal cover edges in star graphs and wheel graphs. Then [7] discusses Hamilton trajectories which focus on simple 4-connected graphs. Furthermore, [2] examines the chromatic number of dual graphs from wheel graphs which will carry out point coloring on dual graphs from wheel graphs. The discussion carried out by previous researchers attracted the author's interest in describing the Hamilton cycle on a wheel graph to develop the article [5] through research that will be carried out in the article.

In the second part, several theoretical bases that support this research are explained. The third part discusses this problem, namely proving that the wheel graph contains the Hamilton cycle. Then continued to the fourth part by explaining the conclusions of the discussion of this article.

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2. PRELIMINARIES

2.1 Graph

Graphs are a branch of mathematics used to represent discrete objects and the relationships between these objects. The visual representation of a graph is to represent objects as points, while the relationships between objects are expressed as edges. According to Aldous et al. [1, p. 26] and Clark et al. [6, p. 13] the definition of a graph is as follows.

Definition 2.1 A graph consists of a set elements called vertices and a set of elements called edges. Each edge joins two vertices.

Definition 2.2 An edge e of a graph G is said to be incident with the vertex v if v is an end vertex of e. In this case we also say that v is incident with e. Two edges e and f which are incident with a common vertex v are said to be adjacent.

According to Munir et al. [9, p. 5] there are several points contained in a set of points $V(G)$ on the graph $G = (V, E)$, the number can be expressed in the notation $|V(G)|$ and several edges contained in a set of edges $E(G)$ in the graph $G = (V, E)$, the number can be expressed in the notation $|E(G)|$. Some references write $|V(G)|$ in the notation form $|V|$ and some also write $|E(G)|$ in the notation form $|E|$. The number of points in the graph $G = (V, E)$ is denoted in the form $|V(G)|$ or $|V|$ this is called the order and the number of edges in the graph $G = (V, E)$ is denoted in the form $|E(G)|$ or $|E|$ this is called size.

2.1.1 Simple Graph and Unsimple Graph

According to Panjaitan et al. [11, p. 4], graphs are grouped into several types depending on the grouping point of view. Grouping graphs based on the presence or absence of rings is divided into two types, namely simple graphs and non-simple graphs.

Definition 2.3 Two or more paths connecting the same pair of nodes are called multiple paths and a path connecting a node to itself is called a loop. A graph without loops and double paths is called a simple graph, while a graph that has loops or double paths is called a non-simple graph. Below, Figure 1 (a) is an example of a simple graphs and (b) is an example of a unsimple graph.

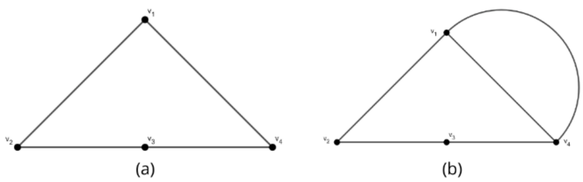

Figure 1. (a) Simple Graph, (b) Unsimple Graph

2.1.2 Directed Graph and Undirected Graph

According to Slamin et al. [12, h. 12], gaphs can be grouped based on the orientation of the edges, namely directed graphs and undirected graphs.

Definition 2.4 A directed graph G is a graph whose edges are not the same length and has a direction where $(u, v) \neq (v, u)$ and a sequence of pairs of vertices must be connected by different edges. Meanwhile, an undirected graph G is a graph whose edges are the same size where $(u, v) = (v, u)$ and the order of pairs of vertices connected by the edges is not taken into account. An example can be seen in Figure 2 (a) is a directed graph and (b) is an undirected graph.

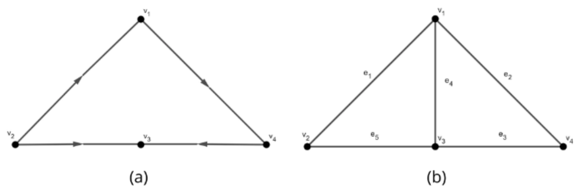

Figure 2. (a) Directed Graph, (b) Undirected Graph

2.2 Degree

The degree of a point is often called the valence of that point. According to Balakrishnan et al. [3, p. 10], the definition of point degree is as follows.

Definition 2.5 Let G be a graph and $v \in V$. The number of edges incident at v in G is called the degree (or valency) of the vertex v in G and is denoted by $d_G(v)$, or simply $d(v)$ when G requires no explicit reference. A loop at v is to be counted twice in computing the degree of v. The minimum (respectively, maximum) of the degrees of the vertices of a graph G is denoted by $\delta(G)$ or δ (respectively, $\Delta(G)$ atau Δ). A graph G is called k-regular if every vertex of G has degree k. A graph is said to be regular if it is k-regular for same nonnegative integer k. In particular, a 3-regular graph is called a cubic graph.

2.3 Subgraph

Subgraphs are defined using the concept of subsets at their points and edges. A graph that has a set of vertices ($V(G)$) and edges ($E(G)$) is a subset of another graph. According to West et al. [13, p. 6], the subgraph is defined as follows.

Definition 2.6 A subgraph of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. We then write $H \subseteq G$ and say that "G contains H".

2.4 Circle Graph

A circle graph is a simple graph in which each point has degree two, denoted by C_n for n points, $n \geq 3$. A circle graph is depicted in the form of a polygon. C_3 can be called a triangle, C_4 a quadrilateral, C_5 a pentagon, C_6 a hexagon and in general C_n can also be called an n-sided. The definition of a circle graph according to Munir et al. [10, p. 377] is as follows.

Definition 2.7 A circle graph is a simple graph where every vertex has degree two. A circle graph with n vertices is denoted by C_n . If the points on C_n are $v_1, v_2, \dots, v_n$, then the sides are $(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n)$, and (v_n, v_1). In other words, there is an edge from the last point, v_n, to the first point, v_1. In Figure 3 below is an example of a circle graph.

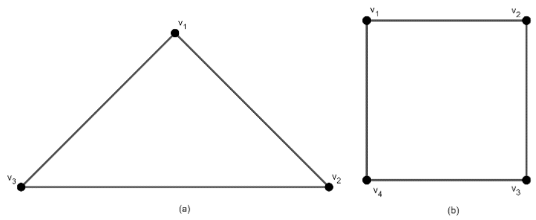

Figure 3. Circle Graph (a) C_3, (b) C_4

2.5 Wheel Graph

The wheel graph which is denoted by W_n for $n \geq 3$ is a graph obtained by adding a new point, namely the center point to the circle graph C_n such that every point on the circle graph C_n is adjacent to the new point. According to Guo [8, p. 1], the definition of the wheel graph is as follows.

Definition 2.8 A wheel graph W_n is a graph with n vertices ($n > 3$), formed by connecting a single vertex to all vertices of an $(n-1)$-cycle. In Figure 4 below is an example of a wheel graph.

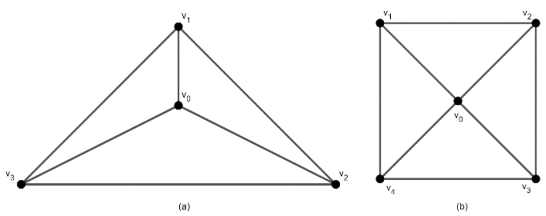

Figure 4. Wheel Graph (a) W_3, (b) W_4

2.6 Hamiltonian

A graph that has a cycle that contains all the points in the graph is called a Hamilton graph. The cycle of a graph that contains each point with the same starting point as the ending point is called a Hamilton cycle, thus a Hamilton graph is a graph that has a Hamilton cycle. In Clark et al. [6, p. 99 - 100] presents the following definitions.

Definition 2.9 A Hamiltonian path in a graph G is a path which contains every vertex of G.

Definition 2.10 A Hamiltonian cycle (or Hamiltonian circuit) in a graph G is a cycle which contains every vertex of G.

3. RESULTS AND DISCUSSION

In this chapter the author discusses the proof that the wheel graph denoted by W_n contains the Hamilton cycle. Then to prove the W_n graph wheel, if you have m where m is the number of W_n that replaces each point in W_n, then it can be denoted as W_n^m which also contains the Hamilton cycle. Then the author forms a general pattern of the Hamilton cycle on the W_n wheel graph and the W_n^m wheel graph which is displayed in the form of an image with the help of a mathematical software, namely Geogebra.

3.1 Hamilton Cycle on the Wheel Graph W_n

The author constructs steps to prove that the wheel graph denoted by W_n contains the Hamilton cycle. For this proof, the author only takes the value $n = 6, 7, 8, 9$, and 10 only. This is because for a wheel graph with $n \geq$ it must contain a Hamilton cycle, so the author takes the values $n \geq 6$ to $n \leq 10$. The proof will be carried out by determining the points and edges on the wheel graph W_n so that the order and size of W_n are obtained. Then look for the trajectory and Hamilton cycle to get a general pattern in the W_n wheel graph.

3.1.1 Hamilton Cycle on the Wheel Graph W_6

It will be shown that the wheel graph contains the Hamilton cycle for $n = 6$. The author uses one of the mathematical software, namely Geogebra, to represent W_6 in the following image.

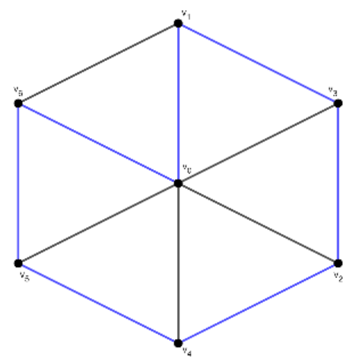

Figure 5. Wheel Graph W_6 and the Hamilton cycle which is colored blue

In Figure 5, it is found that the W_6 wheel graph contains the following set of points and edges.
$V(G) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$,
$E(G) = \{(v_0, v_1), (v_0, v_2), (v_0, v_3), (v_0, v_4), (v_0, v_5), (v_0, v_6), (v_1, v_2), (v_1, v_6), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6)\}$.
The wheel graph W_6 has 7 vertices so the order of G is $|V| = 7$ and has 12 edges so the size of G is $|E| = 12$, contains Hamilton paths $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$ and Hamilton cycles paths $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_0\}$.

3.1.2 Hamilton Cycle on the Wheel Graph W_7

It will be shown that the wheel graph contains the Hamilton cycle for $n = 7$. The author uses one of the mathematical software, namely Geogebra, to represent W_7 in the following image.

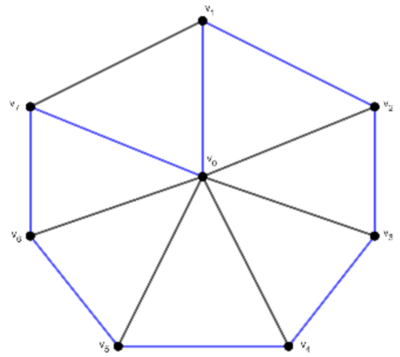

Figure 6. Wheel Graph W_7 and the Hamilton cycle which is colored blue

In Figure 6, it is found that the W_7 wheel graph contains the following set of points and edges.
$V(G) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$,
$E(G) = \{(v_0, v_1), (v_0, v_2), (v_0, v_3), (v_0, v_4), (v_0, v_5), (v_0, v_6), (v_0, v_7), (v_1, v_2), (v_1, v_7), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_7)\}$.
The wheel graph W_7 has 8 vertices so the order of G is $|V| = 8$ and has 14 edges so the size of G is $|E| = 14$, contains Hamilton paths $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and Hamilton cycles paths $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_0\}$.

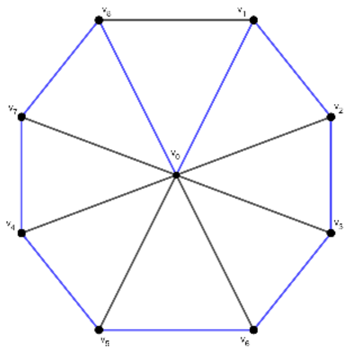

Figure 7. Wheel Graph W_8 and the Hamilton cycle which is colored blue

In Figure 7, it is found that the W_8 wheel graph contains the following set of points and edges.
$V(G) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$,
$E(G) = \{(v_0, v_1), (v_0, v_2), (v_0, v_3), (v_0, v_4), (v_0, v_5), (v_0, v_6), (v_0, v_7), (v_0, v_8), (v_1, v_2), (v_1, v_8), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_7), (v_7, v_8)\}$.

The wheel graph W_8 has 9 vertices so the order of G is $|V| = 9$ and has 16 edges so the size of G is $|E| = 16$, contains Hamilton paths $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and Hamilton cycles paths $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_0\}$.

3.1.3 Hamilton Cycle on the Wheel Graph W_9

It will be shown that the wheel graph contains the Hamilton cycle for $n = 9$. The author uses one of the mathematical software, namely Geogebra, to represent W_9 in the following image.

3.1.4 Hamilton Cycle on the Wheel Graph W_8

It will be shown that the wheel graph contains the Hamilton cycle for $n = 8$. The author uses one of the mathematical software, namely Geogebra, to represent W_8 in the following image.

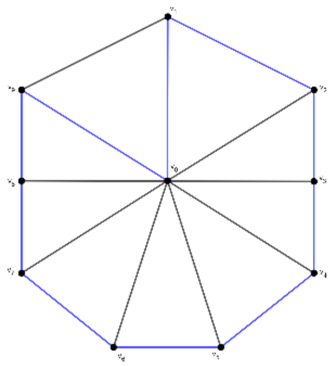

Figure 8. Wheel Graph W_9 and the Hamilton cycle which is colored blue

In Figure 8, it is found that the W_9 wheel graph contains the following set of points and edges.
$V(G) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$,
$E(G) = \{(v_0, v_1), (v_0, v_2), (v_0, v_3), (v_0, v_4), (v_0, v_5), (v_0, v_6), (v_0, v_7), (v_0, v_8), (v_0, v_9), (v_1, v_2), (v_1, v_9), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_7), (v_7, v_8), (v_8, v_9)\}$.

The wheel graph W_9 has 10 vertices so the order of G is $|V| = 10$ and has 18 edges so the size of G is $|E| = 18$, contains Hamilton paths $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and Hamilton cycles paths $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_0\}$.

3.1.5 Hamilton Cycle on the Wheel Graph W_{10}

It will be shown that the wheel graph contains the Hamilton cycle for $n = 10$. The author uses one of the mathematical software, namely Geogebra, to represent W_{10} in the following image.

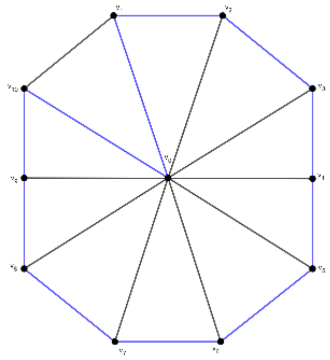

Figure 9. Wheel Graph W_{10} and the Hamilton cycle which is colored blue

In Figure 9, it is found that the W_{10} wheel graph contains the following set of points and edges.
$V(G) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$,
$E(G) = \{(v_0, v_1), (v_0, v_2), (v_0, v_3), (v_0, v_4), (v_0, v_5), (v_0, v_6), (v_0 , v_7), (v_0, v_8), (v_0, v_9), (v_0, v_{10}), (v_1, v_2), (v_1, v_{10}), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_7), (v_7, v_8), (v_8, v_9), (v_9, v_{10})\}$.
The wheel graph W_{10} has 11 vertices so the order of G is $|V| = 11$ and has 20 edges so the size of G is $|E| = 20$, contains Hamilton paths $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ and Hamilton cycles paths $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_0\}$.

In the discussion above, [14] which discusses line graphs in wheel graphs states that wheel graphs have $n + 1$ vertices and $2n$ edges which are expressed in the form of the following theorem.

Theorem 3.1 *The wheel graph W_n has $n + 1$ vertices and $2n$ edges.*

PROOF. Because the wheel graph W_n has n vertices on the outer cycle and 1 vertex at the center point+$|V| = n + 1$. In addition, because the wheel graph W_n has n vertices on the outer cycle, the number of edges on the outer cycle is n and because all the vertices if the outer cycle is directly connected to the center point then there are n more sides, so $|E| = n + n = 2n$. □

Furthermore, in the discussion in this subchapter, conclusions have been drawn from construction to prove that the wheel graph W_n contains the Hamilton cycle which will be further proven using the following theorem.

Theorem 3.2 *For $n \geq 3$ there is a Hamilton cycle in the wheel graph W_n with an arbitrary endpoint $(v, w \in V(W_n))$.*

PROOF. Let $\big(V(W_n)\big) = \{v_0, v_1, v_2 , \dots, v_n\}$. Since W_n is a graph wheel means $(v_i, v_j \in E(W_n)\ \forall\ i, j = 0, 1,2, \dots, n)$, then:

$$\begin{gathered}(v_0, v_1) \in E(W_n),\\ (v_1, v_2) \in E(W_n),\\ \vdots\\ (v_{n-1}, v_n) \in E(W_n),\\ (v_n, v_0) \in E(W_n).\end{gathered}$$

Therefore, there is a cycle that passes through all points in $V(W_n)$ namely $(v_0, v_1), (v_1, v_2), \dots, (v_{n-1}, v_n), (V_n, v_0)$. So, the wheel graph W_n for $n \geq 3$ contains the Hamilton cycle and it can be said that Wn is a Hamilton graph. □

3.2 Hamilton Cycle on the Wheel Graph W_n^m

Construct the steps to prove that if the graph wheel W_n has m where m is the number of W_n that replaces each point in W_n then it can be denoted by W_n^m and also contains the Hamilton cycle. In addition, n in the W_n^m wheel graph is the number of outermost points of W_n^m added to 1 point located in the center.

For example, we will taken $= 2k - 1$, namely $n = 3$ and $n = 2k$, namely $n = 4$, where k is an integer. W_n wheel graph with $n = 3$ or $n = 4$, if it has $m = 1, 2, 3$ where m is the number of W_3 or W_4 that replaces each point in W_3 or W_4 then it can be denoted as W_3^1, W_3^2, W_3^3 or W_4^1, W_4^2, W_4^3 . Then, $n = 3$ or n $n = 4$ in the wheel graph W_3^1, W_3^2, W_3^3 or W_4^1, W_4^2, W_4^3 is the number of outermost points from W_3^1, W_3^2, W_3^3 or W_4^1, W_4^2, W_4^3 which is added to 1 point located in the center. It will be shown in image form with the help of one of the mathematical software, namely Geogebra, so that it is easier to see the changes from W_3 to W_3^1 to W_3^2 to W_3^3 or from W_4 to W_4^1 to W_4^2 to W_4^3 as follows.

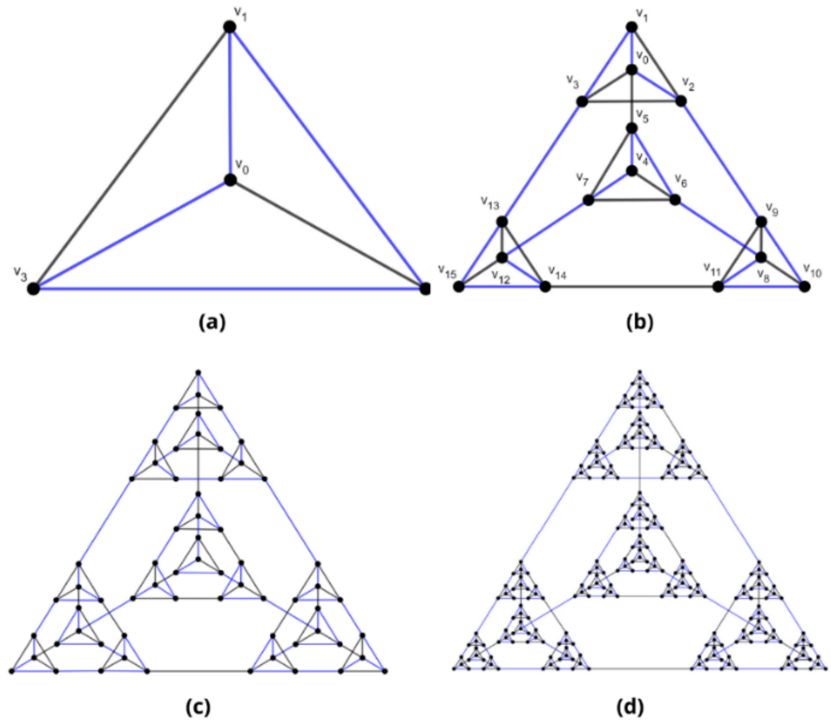

Figure 10. Wheel Graph (a) W_1, (b) W_3^1, (c) W_3^2, (d) W_3^3 and the Hamilton cycle which is colored blue

In Figure 9 and Figure 10 we can see the Hamilton cycle for each graph wheel. However, the wheel graph W_n^m only draws perfectly for $n = 2k$ where k is an integer. This is because in Figure 9 (b), (c), and (d) there are three edges that collide with each other in each wheel graph image.

Next, we will prove the Hamilton cycle on the wheel graph for W_n^m with $n = 6$ and $m = 1, 2,$ and 3. The proof will be carried out by determining the points and edges on the wheel graph W_n^m to obtain the order and size of W_n^m. Then look for the trajectories and Hamilton cycles to get a general pattern in the W_n^m wheel graph.

3.2.1 Hamilton Cycle on the Wheel Graph W_6^1

The W16 wheel graph is a W_6^1 wheel graph where each point is replaced by a W_6 wheel graph. It will be shown that the wheel graph contains the Hamilton cycle for $n = 6$ and $m = 1$. The author uses one of the mathematical software, namely Geogebra, to represent W_6^1 in the following image.

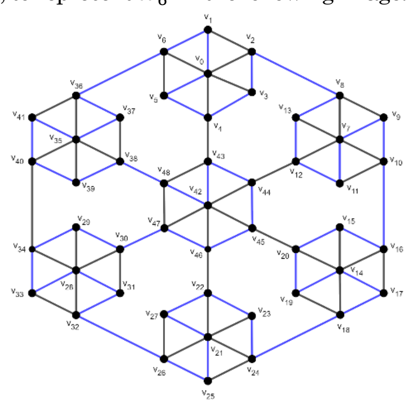

Figure 12. Wheel Graph W_6^1 and the Hamilton cycle which is colored blue

In Figure 12, it is found that the W_6^1 wheel graph contains the following set of points and edges.

$V(G) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}, v_{30}, v_{31}, v_{32}, v_{33}, v_{34}, v_{35}, v_{36}, v_{37}, v_{38}, v_{39}, v_{40}, v_{41}, v_{42}, v_{43}, v_{44}, v_{45}, v_{46}, v_{47}, v_{48}\}$,

$E(G) = \{(v_0, v_1), (v_0, v_2), (v_0, v_3), (v_0, v_4), (v_0, v_5), (v_0, v_6), (v_1, v_2), (v_1, v_6), (v_2, v_3), (v_2, v_8), (v_3, v_4), (v_4, v_5), (v_4, v_{43}), (v_5, v_6), (v_6, v_{36}), (v_7, v_8), (v_7, v_9), (v_7, v_{10}), (v_7, v_{11}), (v_7, v_{12}), (v_7, v_{13}), (v_8, v_9), (v_8, v_{13}), (v_9, v_{10}), (v_{10}, v_{11}), (v_{10}, v_{16}), (v_{11}, v_{12}), (v_{12}, v_{13}), (v_{12}, v_{44}), (v_{14}, v_{15}), (v_{14}, v_{16}), (v_{14}, v_{17}), (v_{14}, v_{18}),$

$(v_{14}, v_{19}), (v_{14}, v_{20}), (v_{15}, v_{16}), (v_{15}, v_{20}), (v_{16}, v_{17}), (v_{17}, v_{18}), (v_{18}, v_{19}), (v_{18}, v_{24}), (v_{19}, v_{20}), (v_{20}, v_{45}), (v_{21}, v_{22}), (v_{21}, v_{23}), (v_{21}, v_{24}), (v_{21}, v_{25}), (v_{21}, v_{26}), (v_{21}, v_{27}), (v_{22}, v_{23}), (v_{22}, v_{27}), (v_{22}, v_{46}), (v_{23}, v_{24}), (v_{24}, v_{25}), (v_{25}, v_{26}), (v_{26}, v_{27}), (v_{26}, v_{32}), (v_{28}, v_{29}), (v_{28}, v_{30}), (v_{28}, v_{31}), (v_{28}, v_{32}), (v_{28}, v_{33}), (v_{28}, v_{34}), (v_{29}, v_{30}), (v_{29}, v_{34}), (v_{30}, v_{31}), (v_{30}, v_{47}), (v_{31}, v_{32}), (v_{32}, v_{33}), (v_{33}, v_{34}), (v_{34}, v_{40}), (v_{35}, v_{36}), (v_{35}, v_{37}), (v_{35}, v_{38}), (v_{35}, v_{39}), (v_{35}, v_{40}), (v_{35}, v_{41}), (v_{36}, v_{37}), (v_{36}, v_{41}), (v_{37}, v_{38}), (v_{38}, v_{39}), (v_{38}, v_{48}), (v_{39}, v_{40}), (v_{40}, v_{41}), (v_{42}, v_{43}), (v_{42}, v_{44}), (v_{42}, v_{45}), (v_{42}, v_{46}), (v_{42}, v_{47}), (v_{42}, v_{48}), (v_{43}, v_{44}), (v_{43}, v_{48}), (v_{44}, v_{45}), (v_{45}, v_{46}), (v_{46}, v_{47}), (v_{47}, v_{48})\}$.

The wheel graph W_6^1 has 49 vertices so the order of G is $|V| = 49$ and has 96 edges so the size of G is $|E| = 96$, contains Hamilton paths $\{v_1, v_0, v_5, v_4, v_3, v_2, v_8, v_{13}, v_{12}, v_{11}, v_7, v_9, v_{10}, v_{16}, v_{15}, v_{20}, v_{19}, v_{14}, v_{17}, v_{18}, v_{24}, v_{23}, v_{22}, v_{27}, v_{21}, v_{25}, v_{26}, v_{32}, v_{31}, v_{28}, v_{33}, v_{34}, v_{29}, v_{30}. v_{47}, v_{46}, v_{45}, v_{44}, v_{43}, v_{42}, v_{48}, v_{38}, v_{39}, v_{40}, v_{41}, v_{35}, v_{37}, v_{36}, v_6\}$ and Hamilton cycles paths $\{v_1, v_0, v_5, v_4, v_3, v_2, v_8, v_{13}, v_{12}, v_{11}, v_7, v_9, v_{10}, v_{16}, v_{15}, v_{20}, v_{19}, v_{14}, v_{17}, v_{18}, v_{24}, v_{23}, v_{22}, v_{27}, v_{21}, v_{25}, v_{26}, v_{32}, v_{31}, v_{28}, v_{33}, v_{34}, v_{29}, v_{30}. v_{47}, v_{46}, v_{45}, v_{44}, v_{43}, v_{42}, v_{48}, v_{38}, v_{39}, v_{40}, v_{41}, v_{35}, v_{37}, v_{36}, v_6, v_1\}$.

3.2.2 Hamilton Cycle on the Wheel Graph W_6^2

The W16 wheel graph is a W_6^2 wheel graph where each point is replaced by a W_6^1 wheel graph. It will be shown that the wheel graph contains the Hamilton cycle for $n = 6$ and $m = 2$. The author uses one of the mathematical software, namely Geogebra, to represent W_6^2 in the following image.

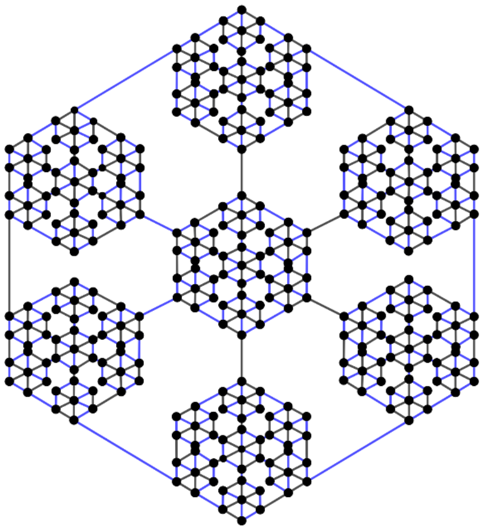

Figure 13. Wheel Graph W_6^2 and the Hamilton cycle which is colored blue

In Figure 13, it is found that the W_6^2 wheel graph contains the following set of points and edges.

$V(G) = \left\{v_0, v_1, v_2, v_3, \dots, v_{((n+m)-(m-1))^{m+1}}\right\}$,

$E(G) = \left\{(v_0, v_1), (v_1, v_2), \dots, \left(v_{((n+m)-(m-1))^{m+1}}, v_0\right)\right\}$.

The wheel graph W_6^2 has 343 vertices so the order of G is $|V| = 343$ and has 684 edges so the size of G is $|E| = 684$, contains Hamilton paths $\left\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, \dots, v_{((n+m)-(m-1))^{m+1}}\right\}$ and Hamilton cycles paths $\left\{v_0, v_1, v_2, v_3, v_4, v_5, v_6 \dots, v_{((n+m)-(m-1))^{m+1}}, v_0\right\}$.

3.2.3 Hamilton Cycle on the Wheel Graph W_6^3

The W16 wheel graph is a W_6^3 wheel graph where each point is replaced by a W_6^2 wheel graph. It will be shown that the wheel graph contains the Hamilton cycle for $n = 6$ and $m = 3$. The author uses one of the mathematical software, namely Geogebra, to represent W_6^3 in the following image.

In Figure 14, it is found that the W_6^3 wheel graph contains the following set of points and edges.

$V(G) = \left\{v_0, v_1, v_2, v_3, \dots, v_{((n+m)-(m-1))^{m+1}}\right\}$,

$E(G) = \left\{(v_0, v_1), (v_1, v_2), \dots, \left(v_{((n+m)-(m-1))^{m+1}}, v_0\right)\right\}$.

The wheel graph W_6^3 has 2401 vertices so the order of G is $|V| = 2401$ and has 4800 edges so the size of G is $|E| = 4800$, contains Hamilton paths $\left\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, \dots, v_{((n+m)-(m-1))^{m+1}}\right\}$ and Hamilton cycles paths $\left\{v_0, v_1, v_2, v_3, v_4, v_5, v_6 \dots, v_{((n+m)-(m-1))^{m+1}}, v_0\right\}$.

Theorem 3.3 *There is a Hamilton cycle in the wheel graph W_n^m where m is the number of W_n that replaces each point in W_n and n is the number of outermost points of W_n^m added to 1 point located in the center.*

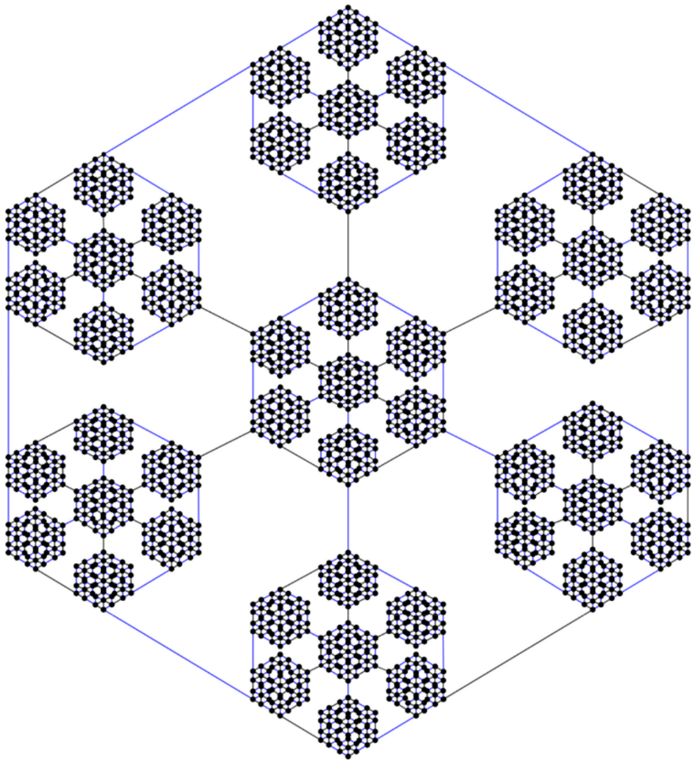

Figure 14. Wheel Graph W_6^3 and the Hamilton cycle which is colored blue

PROOF. For the wheel graph denoted by (W_n^m) the pattern of the set of points $(V(G))$ and edges $(E(G))$ is obtained as follows.

$$V(G) = \left\{v_0, v_1, v_2, v_3, \dots, v_{((n+m)-(m-1))^{m+1}}\right\},$$
$$E(G) = \left\{(v_0, v_1), (v_1, v_2), \dots, \left(v_{((n+m)-(m-1))^{m+1}}, v_0\right)\right\}.$$

Therefore, it is proven that side e_i which is exampled by (v_0, v_1), is directly connected to side e_j which is exampled by (v_1, v_2), for $i \neq j$ and $1 \leq i, j \leq n$. Therefore, the Hamilton cycle for the wheel graph W_n^m is a graph formed from the wheel graph W_n or can be written as $(W_n^m) \cong (W_n)$. □

4. CONCLUSIONS

Based on the discussion, it is known that the wheel graph denoted by W_n contains the Hamilton cycle. Furthermore, the wheel graph is denoted by W_n, if it has m where m is the number of W_n that replaces each point in W_n, then it can be denoted by W_n^m and also contains the Hamilton cycle. For n in the wheel graph Wmn is the number of outermost points of W_n^m added to 1 point located in the center. However, in reality the wheel graph W_n^m only draws perfectly for $n = 2k$ where k is an integer. This is because there are colliding edges for $n = 2k - 1$ where k is an integer.

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