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Stability Results for a Singular System of Generalized p-Fisher-Kolmogoroff Steady State Type

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Abstract

In the present paper, we are interested in the study of the stability results of nontrivial positive weak solutions for the generalized p-Fisher-Kolmogoroff nonlinear steady state problem involving the singular p-Laplacian operator $-\boldsymbol{d}\,\boldsymbol{div}[|\boldsymbol{x}|^{-\boldsymbol{ap}}\,|\boldsymbol{\nabla u}|^{\boldsymbol{p}-2}\boldsymbol{\nabla u}] = \boldsymbol{km}(\boldsymbol{x})|\boldsymbol{x}|^{-(\boldsymbol{a}+1)\boldsymbol{p}+\boldsymbol{c}}\,\boldsymbol{u}[\boldsymbol{\nu} - \boldsymbol{\upsilon u}]$ $\boldsymbol{in}\ \boldsymbol{\Omega}, \boldsymbol{Bu} = \mathbf{0}\ \boldsymbol{on}\ \boldsymbol{\partial\Omega}$, *where* $\boldsymbol{\Omega} \subset \mathbb{R}^{\boldsymbol{n}}$ *is a bounded domain with smooth boundary* $\boldsymbol{Bu} = \boldsymbol{\delta h}(\boldsymbol{x})\boldsymbol{u} + (\mathbf{1}-\boldsymbol{\delta})\frac{\partial u}{\partial n}$ *where* $\boldsymbol{\delta} \in [\mathbf{0},\mathbf{1}], \boldsymbol{h}: \boldsymbol{\partial\Omega} \to \mathbb{R}^{+}$ *with* $\boldsymbol{h} = \mathbf{1}$ *when* $\boldsymbol{\delta} = \mathbf{1}, \mathbf{0} \in \boldsymbol{\Omega}, \mathbf{2} \le \boldsymbol{p} < \mathbf{3}, \mathbf{0} \le \boldsymbol{a} < (\boldsymbol{n} - \boldsymbol{p}/\boldsymbol{p})$, $\boldsymbol{c}$ *is positive parameter, the continuous function,* $\boldsymbol{m}(\boldsymbol{x}): \boldsymbol{\Omega} \to \mathbb{R}$ *satisfies either* $\boldsymbol{m}(\boldsymbol{x}) > \mathbf{0}$ *or* $\boldsymbol{m}(\boldsymbol{x}) < \mathbf{0}$ *for all* $\boldsymbol{x} \in \boldsymbol{\Omega}$, $\boldsymbol{d}, \boldsymbol{k}, \boldsymbol{\nu}$ *are positive parameters and* $\boldsymbol{\upsilon}$ *is nonnegative parameter. We provide a simple proof to establish that every positive solution is stable (unstable) under some certain conditions.*

Keywords: *Stability, Fisher-Kolmogoroff steady state type, singular p-Laplacian*

1. INTRODUCTION

In this paper, we investigate the stability results of positive weak solutions for the generalized singular p-Fisher-Kolmogoroff nonlinear steady state problem

$$\left.\begin{aligned} -d\,div[|x|^{-ap}\,|\nabla u|^{p-2}\nabla u] = km(x)|x|^{-(a+1)p+c}\,u[\nu - \upsilon u]\ \ in\ \Omega, \\ Bu = 0 \text{ on } \partial\Omega \end{aligned}\right\} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $Bu = \delta h(x)u + (1-\delta)\frac{\partial u}{\partial n}$ where $\delta \in [0,1], h: \partial\Omega \to \mathbb{R}^{+}$with $h = 1$ when $\delta = 1, 0 \in \Omega, \mathbf{2} \le \boldsymbol{p} < \mathbf{3}, 0 \le a < (n - p/p)$, c is positive parameter, the continuous function, $m(x): \Omega \to \mathbb{R}$ satisfies either $m(x) > 0$ or $m(x) < 0$ for all $x \in \Omega$, d, k, ν are positive parameters and υ is nonnegative parameter. We provide a simple proof to establish that every positive solution is stable (unstable) under some certain conditions. System (1) is the generalized singular p-Fisher-Kolmogoroff nonlinear steady state problem, where d is the diffusion coefficient, k is the linear reproduction rate and u is the population density.

Problem (1) arises from the population biology of one species. Situations where d is space-dependent are arising in more and more modeling situations of biomedical importance from diffusion of genetically engineered organisms in heterogeneous environments to the effect of white and grey matter in the growth and spread of brain tumours. (see [1-4]).

On the other hand, systems of type (1) have received considerable attention in the last decade (see, e.g., [5,6] and the references therein). It has been shown that for some certain values of ν, υ, system (1) has a rich mathematical structure. In [7,8], system (1) is considered under the hypothesis $|x|^{-ap} = (k/d) = 1, p = 2$ and $f(u) = u$. This corresponds to the Emden-Fowler steady state problem of polytropic index of order one [5]. While in [6,9], system (1) is considered under the hypothesis $|x|^{-ap} = (k/d) = 1, p = 2$ and $f(u) = u(1-u)$, where u is the population density of degree two. This corresponds to the Logistic nonlinear steady state problem. Due to the appearance of singular p-Laplacian operator in (1) and the particular cases; the extensions are challenging and nontrivial.

Many authors are interested in the study of stability and instability of nonnegative solutions of linear [10,11], semilinear [12-16], semipositone [17,18], nonlinear [19-21] and singular systems [22], due to the great number of applications in autocatalytic reaction, in population dynamics, in reaction-diffusion problems, in temperature on plasma, etc.; see [7,23] and references therein. Also, in the recent past, many authors devoted their attention to study the singular p-Laplacian nonlinear systems (see [24-28]).

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Tertikas in [15] has proved the stability and instability results of positive solutions for the semilinear system

$$\left.\begin{aligned} -\Delta u = \lambda f(u) \;\; in\ \Omega, \\ Bu = 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \tag{2}$$

under various choices of the function f. In [17], the authors have been studied the uniqueness and stability of nonnegative solutions for classes of nonlinear elliptic Dirichlet problems in a ball, when the nonlinearity is monotone, negative at the origin, and either concave or convex. In the case $|x|^{-ap} = m(x) = 1, p = 2$ and a function $\lambda u^{\alpha} + u^{\beta}$ instead of $\lambda f(u)$, system (1) have been studied by several authors (see [14,18,27]).

Khafagy in [20] have been studied the stability and instability of nontrivial positive weak solution for the nonlinear system

$$\left.\begin{aligned} -\Delta_{P,p} u + a(x)|u|^{p-2}u = \lambda b(x)u^{\alpha} \;\; in\ \Omega, \\ Bu = 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{3}$$

where $\Delta_{P,p}u$ with $p > 1$ and $P = P(x)$ is a weight function, denotes the weighted p-Laplacian defined by $\Delta_{P,p}u = div[P(x)|\nabla u|^{p-2}\nabla u]$, $a(x)$ is a weight function, the continuous function $b(x): \Omega \to \mathbb{R}$ satisfies either $b(x) > 0$ or $b(x) < 0$ for all $x \in \Omega$, λ is a positive parameter, $0 < \alpha < p - 1$ and Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $Bu = \delta h(x)u + (1-\delta)\frac{\partial u}{\partial n}$, where $\delta \in [0,1]$, $h: \partial\Omega \to \mathbb{R}^{+}$ with $h = 1$ when $\delta = 1$. He proved that if $0 < \alpha < p - 1$ and $b(x) > 0 (< 0)$ for all $x \in \Omega$, then every positive weak solution u of (3) is linearly stable (unstable) respectively.

We recall that, if u be any positive weak solution of (1), then the linearized equation of (1) about u is given by

$$\left.\begin{aligned} -(p-1)div[|x|^{-ap}\,|\nabla u|^{p-2}\nabla\varphi] - (k/d)m(x)|x|^{-(a+1)p+c} f_u\varphi = \mu\varphi, \qquad x \in \Omega, \\ B\varphi = 0 \qquad\qquad \text{on } \partial\Omega, \end{aligned}\right\} \tag{4}$$

where μ is the eigenvalue corresponding to the eigenfunction φ.

Definition 1. 1. [19] A solution u of (1) is called stable solution if all eigenvalues of (4) are strictly positive, which can be implied if the principal eigenvalue $\mu_1 > 0$. Otherwise u is unstable.

2. MAIN RESULTS

Our main goal of this section is to prove the stability and instability of the positive weak solution u of (1). Our main results are formulated in the following theorems.

Theorem 2. 1. If $m(x) \geq 0$ for all $x \in \Omega$, then every positive weak solution of (1) is linearly stable.

PROOF. Let u_0 be any positive weak solution of (1), then the linearized equation bout u_0 is

$$\left.\begin{aligned} -(p-1)div[|x|^{-ap}\,|\nabla u_0|^{p-2}\nabla\varphi] - (k/d)m(x)|x|^{-(a+1)p+c}(\nu - 2\nu u_0)\varphi = \mu\varphi, \qquad x \in \Omega, \\ B\varphi = 0 \text{ on } \partial\Omega. \end{aligned}\right\} \tag{5}$$

Let μ_1 be the first eigenvalue of (5) and let $\psi > 0$ be the corresponding eigenfunction. Multiplying (5) by $(-u_0)$ and integrating over Ω, we have

$$(p-1)\int_{\Omega} u_0\, div[|x|^{-ap}\,|\nabla u_0|^{p-2}\nabla\psi]dx \\ +(k/d)\int_{\Omega} m(x)\,|x|^{-(a+1)p+c}\,u_0(\nu - 2\nu u_0)\psi dx = -\mu_1\int_{\Omega} u_0\,\psi dx\,. \tag{6}$$

order and the number of edges in the graph $G = (V, E)$ is denoted in the form $|E(G)|$ or $|E|$ this is called size.

The first term of the L.H.S. of (6) may be written in the form

$$\int_\Omega u_0\, div[|x|^{-ap}\, |\nabla u_0|^{p-2}\nabla\psi]dx = \int_\Omega u_0\, [|x|^{-ap}\, |\nabla u_0|^{p-2}]\nabla\cdot\nabla\psi dx + \int_\Omega u_0\, \nabla\psi\nabla[|x|^{-ap}\, |\nabla u_0|^{p-2}]dx.$$

Using Green's first identity, one has

$$\int_\Omega u_0\, div[|x|^{-ap}\, |\nabla u_0|^{p-2}\nabla\psi]dx = -\int_\Omega \nabla[u_0|x|^{-ap}\, |\nabla u_0|^{p-2}]\nabla\psi\, dx$$

$$+\int_\Omega u_0\, \nabla[|x|^{-ap}\, |\nabla u_0|^{p-2}]\nabla\psi dx + \int_{\partial\Omega} u_0\, [|x|^{-ap}\, |\nabla u_0|^{p-2}]\frac{\partial\psi}{\partial n}ds.$$

So,

$$\begin{aligned}\int_\Omega u_0\, div[|x|^{-ap}\, |\nabla u_0|^{p-2}\nabla\psi]dx \\ = -\int_\Omega [|x|^{-ap}\, |\nabla u_0|^{p-2}]\nabla u_0\nabla\psi\, dx \\ + \int_{\partial\Omega} u_0\, [|x|^{-ap}\, |\nabla u_0|^{p-2}]\frac{\partial\psi}{\partial n}\, ds.\end{aligned} \tag{7}$$

From (7) in (6), we obtain

$$\begin{aligned}(p-1)\int_{\partial\Omega} u_0\, [|x|^{-ap}\, |\nabla u_0|^{p-2}]\frac{\partial\psi}{\partial n}ds - (p-1)\int_\Omega [|x|^{-ap}\, |\nabla u_0|^{p-2}]\nabla u_0\nabla\psi\, dx \\ + (k/d)\int_\Omega m(x)\, |x|^{-(a+1)p+c}\, u_0(\nu - 2\upsilon u_0)\psi dx \\ = -\mu_1 \int_\Omega u_0\, \psi dx.\end{aligned} \tag{8}$$

Multiplying (1) by ψ and integrating over Ω, we have

$$\begin{aligned}(k/d)\int_\Omega m(x)\, |x|^{-(a+1)p+c}\, u_0(\nu - \upsilon u_0)\psi dx \\ = -\int_\Omega \psi\, div[|x|^{-ap}\, |\nabla u_0|^{p-2}\nabla u_0]dx.\end{aligned} \tag{9}$$

The R.H.S. of (9) may be written in the form

$$\int_\Omega div[|x|^{-ap}\, |\nabla u_0|^{p-2}\nabla u_0]\psi\, dx$$

$$= \int_\Omega \nabla u_0\nabla[|x|^{-ap}\, |\nabla u_0|^{p-2}\nabla u_0]\psi\, dx + \int_\Omega \psi\, [|x|^{-ap}\, |\nabla u_0|^{p-2}\nabla u_0]\nabla\cdot(\nabla u_0)dx.$$

Applying Green's first identity, we have

$$\int_{\Omega} div[|x|^{-ap}\,|\nabla u_0|^{p-2}\nabla u_0]\psi\,dx$$
$$= \int_{\Omega} \nabla u_0 \nabla[|x|^{-ap}\,|\nabla u_0|^{p-2}\nabla u_0]\psi\,dx - \int_{\Omega} \nabla[\psi|x|^{-ap}\,|\nabla u_0|^{p-2}]\nabla u_0\,dx + \int_{\partial\Omega} \psi\,[|x|^{-ap}\,|\nabla u_0|^{p-2}]\frac{\partial u_0}{\partial n}ds.$$

Hence,

$$\int_{\Omega} div[|x|^{-ap}\,|\nabla u_0|^{p-2}\nabla u_0]\psi\,dx = -\int_{\Omega} [|x|^{-ap}\,|\nabla u_0|^{p-2}]\nabla\psi\nabla u_0\,d + \int_{\partial\Omega} \psi\,[|x|^{-ap}\,|\nabla u_0|^{p-2}]\frac{\partial u_0}{\partial n}ds. \tag{10}$$

From (10) in (9), we have

$$(k/d)\int_{\Omega} m(x)\,|x|^{-(a+1)p+c}\,u_0(\nu - \upsilon u_0)\psi dx = \int_{\Omega} \nabla\psi[|x|^{-ap}\,|\nabla u_0|^{p-2}]\nabla u_0\,dx - \int_{\partial\Omega} \psi\,[|x|^{-ap}\,|\nabla u_0|^{p-2}]\frac{\partial u_0}{\partial n}ds. \tag{11}$$

Multiplying (11) by $(p-1)$and adding with (8), we have

$$(p-1)\int_{\partial\Omega} [|x|^{-ap}\,|\nabla u_0|^{p-2}]\left[u_0\frac{\partial\psi}{\partial n} - \psi\frac{\partial u_0}{\partial n}\right]ds + (k/d)\int_{\Omega} m(x)\,|x|^{-(a+1)p+c}\,[\nu u_0(2-p) + \upsilon u_0^2(p-3)]\psi dx = -\mu_1\int_{\Omega} u_0\,\psi dx. \tag{12}$$

Now, when $\delta = 1$, we have $Bu_0 = u_0 = 0$ for $s \in \partial\Omega$ and, we have $\psi = 0$ for $s \in \partial\Omega$. Then

$$\int_{\partial\Omega} [|x|^{-ap}\,|\nabla u_0|^{p-2}][u_0\frac{\partial\psi}{\partial n} - \psi\frac{\partial u_0}{\partial n}]\,ds = 0. \tag{13}$$

Also, when $\delta \neq 1$, we have

$$\frac{\partial u_0}{\partial n} = -\frac{\delta h u_0}{1-\delta} = \frac{\partial\psi}{\partial n},$$

which confirms the result given by (13). So, (12) becomes

$$(k/d)\int_{\Omega} m(x)\,|x|^{-(a+1)p+c_1}\,[\nu u_0(2-p) + \upsilon u_0^2(p-3)]\psi dx = -\mu_1\int_{\Omega} u_0\,\psi dx. \tag{14}$$

Since $2 \leq p < 3$ and $m(x) > 0$ for all x, then (14) becomes

$$-\mu_1 \int_{\Omega} u_0\, \psi dx < 0. \tag{15}$$

Hence $\mu_1 > 0$ and the result follows. □

Theorem 2. 2. If $m(x) < 0$ for all $x \in \Omega$, then every positive weak solution of (1) is linearly unstable.

PROOF. As in the proof of Theorem 2. 1., we have

$$-\mu_1 \int_{\Omega} u_0\, \psi dx > 0. \tag{16}$$

So $\mu_1 > 0$ and the result follows. □

3. APPLICATIONS AND RELATED RESULTS

To illustrate the value of the results we obtained, we now present the following three examples.

Example 3.1. Consider the singular Emden-Fowler steady-state problem of polytropic index of order one,

$$\left.\begin{array}{c} -div[|x|^{-ap}\,|\nabla u|^{p-2}\nabla u] = \lambda m(x)|x|^{-(a+1)p+c}\, u \ \ in\ \Omega, \\ Bu = 0 \text{ on } \partial\Omega \end{array}\right\} \tag{17}$$

with $m(x) > 0$ for all $x \in \Omega$. Then according to Theorem 2. 1., every positive weak solution of (17) is stable.

Example 3. 2. Consider the singular population density steady-state problem of degree two,

$$\left.\begin{array}{c} -div[|x|^{-ap}\,|\nabla u|^{p-2}\nabla u] = \lambda m(x)|x|^{-(a+1)p+c}\, [u - u^2] \ \ in\ \Omega, \\ Bu = 0 \text{ on } \partial\Omega \end{array}\right\} \tag{18}$$

with $m(x) > 0$ for all $\in \Omega$. Hence according to Theorem 2. 1., every positive weak solution of (18) is stable.

Example 3. 3. Consider the singular chemotaxis steady-state problem of degree two,

$$\left.\begin{array}{c} -div[|x|^{-ap}\,|\nabla u|^{p-2}\nabla u] = \lambda m(x)|x|^{-(a+1)p+c}\, [-u + u^2] \ \ in\ \Omega, \\ Bu = 0 \text{ on } \partial\Omega \end{array}\right\} \tag{19}$$

with $m(x) > 0$ for all $x \in \Omega$. As a sequence of Theorem 2. 2., every positive weak solution of (19) is unstable.

REFERENCES

1. Murray, J., 1989., Mathematical Biology, I. An Introduction, Third Edition, Springer, New York.
2. Khafagy, S., Serag, H., 2022, Stability of positive weak solution for generalized weighted p-Fisher-Kolmogoroff nonlinear stationary-state problem. *European Journal of Mathematical Analysis*, Volume 2 :88.

3. Lan, K., 2017, Population models with quasi-yield harvest rates, *Mathematical Biosciences and Engineering*, Volume 14, pp. 467-490.
4. Khafagy, S., Sadeghi, Z., 2024, Existence of positive weak solution for a weighted system of autocatalytic reaction steady state type, Eur. J. Math. Appl., Volume 4 (11), pp. 1-7.
5. Flores, I., 2004, A resonance phenomenon for ground states of an elliptic equation of Emden-Fowler type, *Jour. Diff. Eqn.,* Volume 198, pp. 1-15.
6. Khafagy, S., Rasouli, S., Serag, H., 2023, Existence results for fractional Fisher-Kolmogoroff steady state problem, Eur. *J. Math. Appl., Volume* 3, pp. 1-7.
7. Shi, J., Zhitao, Z, 2005, Lectures on solution set of semilinear elliptic equations (in Tokyo Metropolitan University).
8. Goddard II, J., Shivaji, R., 2014, Diffusive logistic equation with constant yield harvesting and negative density dependent emigration on the boundary, *J. Math. Anal. Appl.,* Volume 414 (2), pp. 561-573.
9. Lan, K. Q., Wu, J. H., 2003, Travelling wavefronts of scalar reaction–diffusion equations with and without delays, *Nonlinear Analysis: Real World Applications*, Volume 4, pp.173–188.
10. Afrouzi, G. Sadeeghi, Z., 2008, Stability results for a class of elliptic problems, *International Journal of Nonlinear Science*, Volume 6 (2), pp. 114-117.
11. Khafagy, S., Ezzat, A., 2024, Existence and stability of positive weak solutions for a class of chemically reacting systems. *Eur. J. Math. Appl.,* Volume 4 (2), pp. 1-12.
12. Afonso, D.G., Iacopetti, A., Pacella, F., 2024, Energy Stability for a Class of Semilinear Elliptic Problems, *J. Geom. Anal.*, Volume **34** (75), pp. 1-43.
13. Karatson, I., Simon, P., 2001, On the stability properties of nonnegative solutions of semilinear problems with convex or concave nonlinearity, *Journal of Computational and Applied Mathematics*, Volume 131, pp. 497-501.
14. Maya, C. Shivaji, R., 1998, Instability of nonnegative solutions for a class of semilinear elliptic boundary value problems, *J. Comput. Appl. Math.,* Volume 88, pp. 125-128.
15. Tertikas, A., 1992, Stability and instability of positive solutions of semilinear problems, *Proc. Amer. Math. Soc.*, Volume 114, pp. 1035-1040.
16. Voros, I. 2004, Stability properties of nonnegative solutions of semilinear symmetric cooperative systems, *Electronic Jour. Diff. Eqns.,* Volume 2004 (105), pp. 1-6.
17. Ali, I., Castro, A., Shivaji, R., 1993, Uniqueness and stability of nonnegative solutions for semipositone problems in a ball, *Proc. Amer. Math. Soc.,* Volume 117, pp. 775-782.
18. Brown, K., Shivaji, R., 1991, Instability of nonnegative solutions for a class of semipositone problems, *Proc. Amer. Math. Soc.,* Volume 112, pp. 121-124.
19. Afrouzi, G., Rasouli, S., 2006, Stability properties of non-negative solutions to a non-autonomous p-Laplacian equation, *Chaos, Solitons and fractals*, Volume 29, pp. 1095-1099.
20. Khafagy, S., 2015, On the stability of positive weak solution for weighted p-Laplacian nonlinear system, *New Zealand Journal of Mathematics*, Volume 45, pp. 39-43.
21. Khafagy, S., Serag, H., 2020, On the stability of positive weak solution for (p,q)-Laplacian nonlinear system, *Applied Mathematics E-Notes*, Volume 20, pp. 108-114.
22. Khafagy, S., Serag, H., 2018, Stability results of positive weak solution for singular p-Laplacian nonlinear system, *J. Appl. Math. & Informatics,* Volume 36, pp. 173-179.
23. Atkinson, C, Ali., K., 1992, Some boundary value problems for the Bingham model, *J. Non-Newton. Fluid Mech.*, Volume 41, pp. 339-363.
24. Khafagy, S., 2016, Maximum Principle and Existence of Weak Solutions for Nonlinear System Involving Singular p-Laplacian Operators, *J. Part. Diff. Eq.*, Volume 29, pp. 89-101.
25. Xuan, B., 2003, Existence results for a superlinear singular equation of Caffarelli-Kohn-Nirenberg type, *Bolet´ın de Matem´aticas*, Volume X, pp. 47-58.
26. Khafagy, S., Serag, H., 2020, On the existence of positive weak solution for nonlinear system with singular weights, *Journal of Contemporary Math-ematical Analysis (Armenian Academy of Sciences)*, Volume 55 (4), pp. 259-267.
27. Farkas, G. Simon, P., 2001, Stability properties of positive solutions to partial differential equations with delay, *Electron. J. Diff. Eqn.,* Volume 2001 (64), pp. 1-8.

28. Khafagy, S., El-Zahrani, E., Serag, H., 2020, Existence and uniqueness of weak solution for nonlinear weighted (p, q)-Laplacian system with application on an optimal control problem, *Jordan Journal of Mathematics and Statistics (JJMS)*, Volume 15 (4A), pp. 983-998.