Stability Results for a Singular System of Generalized p-Fisher-Kolmogoroff Steady State Type

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Abstract

In the present paper, we are interested in the study of the stability results of nontrivial positive weak solutions for the generalized p-Fisher-Kolmogoroff nonlinear steady state problem involving the singular p-Laplacian operator 
\[-d \, \text{div}[|x|^{-ap} |\nabla u|^{p-2}\nabla u] = km(x)|x|^{-(a+1)p+c} u[v - vu] \text{ in } \Omega,\]
\[Bu = 0 \text{ on } \partial \Omega,\]
where \(\Omega \subset \mathbb{R}^n\) is a bounded domain with smooth boundary \(Bu = \delta h(x)u + (1 - \delta)^h\) where \(h \in [0, 1], h: \partial \Omega \to \mathbb{R}^+\) with \(h = 1\) when \(\delta = 1, 0 \in \Omega, 2 \leq p < 3, 0 \leq a < (n - p)/p, c\) is positive parameter, the continuous function, \(m(x): \Omega \to \mathbb{R}\) satisfies either \(m(x) > 0\) or \(m(x) < 0\) for all \(x \in \Omega, d, k, v\) are positive parameters and \(v\) is nonnegative parameter. We provide a simple proof to establish that every positive solution is stable (unstable) under some certain conditions.

Keywords: Stability, Fisher-Kolmogoroff steady state type, singular p-Laplacian

1. INTRODUCTION

In this paper, we investigate the stability results of positive weak solutions for the generalized singular p-Fisher-Kolmogoroff nonlinear steady state problem

\[-d \, \text{div}[|x|^{-ap} |\nabla u|^{p-2}\nabla u] = km(x)|x|^{-(a+1)p+c} u[v - vu] \text{ in } \Omega,\]
\[Bu = 0 \text{ on } \partial \Omega,\]

where \(\Omega \subset \mathbb{R}^n\) is a bounded domain with smooth boundary \(Bu = \delta h(x)u + (1 - \delta)^h\) where \(\delta \in [0, 1], h: \partial \Omega \to \mathbb{R}^+\) with \(h = 1\) when \(\delta = 1, 0 \in \Omega, 2 \leq p < 3, 0 \leq a < (n - p)/p, c\) is positive parameter, the continuous function, \(m(x): \Omega \to \mathbb{R}\) satisfies either \(m(x) > 0\) or \(m(x) < 0\) for all \(x \in \Omega, d, k, v\) are positive parameters and \(v\) is nonnegative parameter. We provide a simple proof to establish that every positive solution is stable (unstable) under some certain conditions. System (1) is the generalized singular p-Fisher-Kolmogoroff nonlinear steady state problem, where \(d\) is the diffusion coefficient, \(k\) is the linear reproduction rate and \(u\) is the population density.

Problem (1) arises from the population biology of one species. Situations where \(d\) is space-dependent are arising in more and more modeling situations of biomedical importance from diffusion of genetically engineered organisms in heterogeneous environments to the effect of white and grey matter in the growth and spread of brain tumours. (see [1-4]).

On the other hand, systems of type (1) have received considerable attention in the last decade (see, e.g., [5,6] and the references therein). It has been shown that for some certain values of \(v, u\), system (1) has a rich mathematical structure. In [7,8], system (1) is considered under the hypothesis \(|x|^{-ap} = (k/d) = 1, p = 2\) and \(f(u) = u\). This corresponds to the Emden-Fowler steady state problem of polytropic index of order one [5]. While in [6,9], system (1) is considered under the hypothesis \(|x|^{-ap} = (k/d) = 1, p = 2\) and \(f(u) = u(1 - u)\), where \(u\) is the population density of degree two. This corresponds to the Logistic nonlinear steady state problem. Due to the appearance of singular p-Laplacian operator in (1) and the particular cases; the extensions are challenging and nontrivial.

Many authors are interested in the study of stability and instability of nonnegative solutions of linear [10,11], semilinear [12-16], semipositive [17,18], nonlinear [19-21] and singular systems [22], due to the great number of applications in autocatalytic reaction, in population dynamics, in reaction-diffusion problems, in temperature on plasma, etc.; see [7,23] and references therein. Also, in the recent past, many authors devoted their attention to study the singular p-Laplacian nonlinear systems (see [24-28]).

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Multiplying (5) by \( \lambda \) gives
\[
-\Delta u = \lambda f(u) \quad \text{in } \Omega, \\
Bu = 0 \quad \text{on } \partial \Omega
\]
(2)
under various choices of the function \( f \). In [17], the authors have been studied the uniqueness and stability of nonnegative solutions for classes of nonlinear elliptic Dirichlet problems in a ball, when the nonlinearity is monotone, negative at the origin, and either concave or convex. In the case \( |x|^{-ap} f(x) = 1, p = 2 \) and a function \( \lambda u^a + u^\beta \) instead of \( \lambda f(u) \), system (1) have been studied by several authors (see [14,18,27]).

Tertikas in [15] has proved the stability and instability results of positive solutions for the semilinear system
\[
-\Delta u = \lambda f(u) \quad \text{in } \Omega, \\
Bu = 0 \quad \text{on } \partial \Omega
\]
where \( \Delta_p u \) with \( p > 1 \) and \( P = P(x) \) is a weight function, denotes the weighted \( p \)-Laplacean defined by \( \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \), \( a(x) \) is a weight function, the continuous function \( b(x): \Omega \to \mathbb{R} \) satisfies either \( b(x) > 0 \) or \( b(x) < 0 \) for all \( x \in \Omega \), \( \lambda \) is a positive parameter, \( 0 < a < p - 1 \) and \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( Bu = \delta h(x)u + (1 - \delta)^{p-2}u \) where \( \delta \in [0,1], h: \partial \Omega \to \mathbb{R}^* \) with \( h = 1 \) when \( \delta = 1 \). He proved that if \( 0 < a < p - 1 \) and \( b(x) > 0(0 < 0) \) for all \( x \in \Omega \), then every positive weak solution \( u \) of (3) is linearly stable (unstable) respectively.

We recall that, if \( u \) be any positive weak solution of (1), then the linearized equation of (1) about \( u \) is given by
\[
-(p-1)\text{div}[|x|^{-ap} |\nabla u|^{p-2} \nabla \varphi] - (k/d)m(x)|x|^{-(a+1)p+c} f_u \varphi = \mu \varphi, \quad x \in \Omega, \\
B \varphi = 0 \quad \text{on } \partial \Omega
\]
(4)
where \( \mu \) is the eigenvalue corresponding to the eigenfunction \( \varphi \).

**Definition 1.1.** [19] A solution \( u \) of (1) is called stable solution if all eigenvalues of (4) are strictly positive, which can be implied if the principal eigenvalue \( \mu_1 > 0 \). Otherwise \( u \) is unstable.

### 2. MAIN RESULTS

Our main goal of this section is to prove the stability and instability of the positive weak solution \( u \) of (1). Our main results are formulated in the following theorems.

**Theorem 2.1.** If \( m(x) \geq 0 \) for all \( x \in \Omega \), then every positive weak solution of (1) is linearly stable.

**Proof.** Let \( u_0 \) be any positive weak solution of (1), then the linearized equation bout \( u_0 \) is
\[
-(p-1)\text{div}[|x|^{-ap} |\nabla u_0|^{p-2} \nabla \varphi] - (k/d)m(x)|x|^{-(a+1)p+c} (v - 2u_0) \varphi = \mu \varphi, \quad x \in \Omega, \\
B \varphi = 0 \quad \text{on } \partial \Omega.
\]
(5)
Let \( \mu_1 \) be the first eigenvalue of (5) and let \( \psi > 0 \) be the corresponding eigenfunction. Multiplying (5) by \( (-u_0) \) and integrating over \( \Omega \), we have
\[
(p-1) \int_\Omega u_0 \text{div}[|x|^{-ap} |\nabla u_0|^{p-2} \nabla \psi]dx + (k/d) \int_\Omega m(x)|x|^{-(a+1)p+c} u_0(v - 2u_0)\psi dx = -\mu_1 \int_\Omega u_0 \psi dx.
\]
(6)
order and the number of edges in the graph \( G = (V, E) \) is denoted in the form \(|E(G)|\) or \(|E|\) this is called size.

The first term of the L.H.S. of (6) may be written in the form

\[
\int_{\Omega} u_0 \, \text{div} \left[ |x|^{-ap} |\nabla u_0|^{p-2} \nabla \psi \right] dx = \int_{\Omega} u_0 \, |x|^{-ap} |\nabla u_0|^{p-2} |\nabla \psi| dx + \int_{\Omega} u_0 \, \nabla \psi \nabla \left[ |x|^{-ap} |\nabla u_0|^{p-2} \right] dx.
\]

Using Green's first identity, one has

\[
\int_{\Omega} u_0 \, \text{div} \left[ |x|^{-ap} |\nabla u_0|^{p-2} \nabla \psi \right] dx = -\int_{\Omega} \nabla [u_0 \, |x|^{-ap} |\nabla u_0|^{p-2}] \nabla \psi dx + \int_{\partial \Omega} u_0 \, |x|^{-ap} |\nabla u_0|^{p-2} \frac{\partial \psi}{\partial n} ds.
\]

So,

\[
\int_{\Omega} u_0 \, \text{div} \left[ |x|^{-ap} |\nabla u_0|^{p-2} \nabla \psi \right] dx = -\int_{\Omega} |x|^{-ap} |\nabla u_0|^{p-2} \nabla u_0 \nabla \psi dx + \int_{\partial \Omega} u_0 \, |x|^{-ap} |\nabla u_0|^{p-2} \frac{\partial \psi}{\partial n} ds.
\]

From (7) in (6), we obtain

\[
(p - 1) \int_{\partial \Omega} u_0 \, |x|^{-ap} |\nabla u_0|^{p-2} \frac{\partial \psi}{\partial n} ds - (p - 1) \int_{\Omega} |x|^{-ap} |\nabla u_0|^{p-2} \nabla u_0 \nabla \psi dx + (k/d) \int_{\Omega} m(x) |x|^{-(a+1)p+c} u_0 (v - 2\nu u_0) \psi dx
\]

\[
= -\mu_1 \int_{\Omega} u_0 \psi dx.
\]

Multiplying (1) by \( \psi \) and integrating over \( \Omega \), we have

\[
(k/d) \int_{\Omega} m(x) |x|^{-(a+1)p+c} u_0 (v - \nu u_0) \psi dx = -\int_{\Omega} \psi \, \text{div} \left[ |x|^{-ap} |\nabla u_0|^{p-2} \nabla u_0 \right] dx.
\]

The R.H.S. of (9) may be written in the form

\[
\int_{\Omega} \text{div} \left[ |x|^{-ap} |\nabla u_0|^{p-2} \nabla u_0 \right] \psi dx = \int_{\Omega} \nabla u_0 \nabla \left[ |x|^{-ap} |\nabla u_0|^{p-2} \psi \right] dx + \int_{\Omega} \psi \, \left[ |x|^{-ap} |\nabla u_0|^{p-2} \nabla u_0 \right] \nabla \left( \nabla u_0 \right) dx.
\]

Applying Green's first identity, we have
\[
\int_{\Omega} \text{div}([x]^{-ap} |\nabla u_0|^{p-2} \nabla u_0)\psi \, dx = \int_{\Omega} \nabla u_0 \nabla ([x]^{-ap} |\nabla u_0|^{p-2} \nabla u_0) \psi \, dx - \int_{\Omega} \|\psi\|_{x^{-ap}} |\nabla u_0|^{p-2} \nabla u_0 \, dx + \int_{\partial \Omega} \psi \, [\nabla |x|^{-ap} \nabla u_0^{p-2}]_{\partial \Omega} u_0 \, ds.
\]

Hence,

\[
\int_{\Omega} \text{div}([x]^{-ap} |\nabla u_0|^{p-2} \nabla u_0)\psi \, dx = - \int_{\Omega} ([x]^{-ap} |\nabla u_0|^{p-2}) \nabla u_0 \psi \, dx + \int_{\partial \Omega} \psi \, [\nabla |x|^{-ap} \nabla u_0^{p-2}]_{\partial \Omega} u_0 \, ds.
\]

From (10) in (9), we have

\[
(k/d) \int_{\Omega} m(x) |x|^{-(a+1)p+c} u_0 (\nu - \nu u_0) \psi dx = \int_{\Omega} \nabla \psi \, [\nabla |x|^{-ap} |\nabla u_0|^{p-2}] \nabla u_0 \, dx - \int_{\partial \Omega} \psi \, [\nabla |x|^{-ap} |\nabla u_0|^{p-2}]_{\partial \Omega} u_0 \, ds.
\]

Multiplying (11) by \((p - 1)\) and adding with (8), we have

\[
(p - 1) \int_{\partial \Omega} ([x]^{-ap} |\nabla u_0|^{p-2}) \left[ \frac{\partial \phi}{\partial \nu} - \psi \frac{\partial u_0}{\partial \nu} \right] ds + (k/d) \int_{\Omega} m(x) |x|^{-(a+1)p+c} [\nu u_0 (2 - p) + \nu u_0^2 (p - 3)] \psi dx = -\mu_1 \int_{\Omega} u_0 \psi dx.
\]

Now, when \(\delta = 1\), we have \(Bu_0 = u_0 = 0\) for \(s \in \partial \Omega\) and, we have \(\psi = 0\) for \(s \in \partial \Omega\). Then

\[
\int_{\partial \Omega} ([x]^{-ap} |\nabla u_0|^{p-2}) \left[ u_0 \frac{\partial \phi}{\partial \nu} - \psi \frac{\partial u_0}{\partial \nu} \right] ds = 0.
\]

Also, when \(\delta \neq 1\), we have

\[
\frac{\partial u_0}{\partial \nu} = - \frac{\delta h u_0}{1 - \delta} = \frac{\partial \psi}{\partial \nu},
\]

which confirms the result given by (13). So, (12) becomes

\[
(k/d) \int_{\Omega} m(x) |x|^{-(a+1)p+c} [\nu u_0 (2 - p) + \nu u_0^2 (p - 3)] \psi dx = -\mu_1 \int_{\Omega} u_0 \psi dx.
\]

Since \(2 \leq p < 3\) and \(m(x) > 0\) for all \(x\), then (14) becomes
\[-\mu \int_\Omega u_0 \psi \, dx < 0. \]

Hence \( \mu_1 > 0 \) and the result follows.

**Theorem 2.2.** If \( m(x) < 0 \) for all \( x \in \Omega \), then every positive weak solution of (1) is linearly unstable.

**Proof.** As in the proof of Theorem 2.1., we have

\[-\mu \int_\Omega u_0 \psi \, dx > 0. \]

So \( \mu_1 > 0 \) and the result follows.

### 3. APPLICATIONS AND RELATED RESULTS

To illustrate the value of the results we obtained, we now present the following three examples.

**Example 3.1.** Consider the singular Emden-Fowler steady-state problem of polytropic index of order one,

\[-\text{div}(|x|^{-a} \nabla u) + p \nabla u = \lambda m(x)|x|^{-(a+1)p+c} u \text{ in } \Omega, \]

\[Bu = 0 \text{ on } \partial \Omega, \]

with \( m(x) > 0 \) for all \( x \in \Omega \). Then according to Theorem 2.1., every positive weak solution of (17) is stable.

**Example 3.2.** Consider the singular population density steady-state problem of degree two,

\[-\text{div}(|x|^{-a} \nabla u) + p \nabla u = \lambda m(x)|x|^{-(a+1)p+c} [u - u^2] \text{ in } \Omega, \]

\[Bu = 0 \text{ on } \partial \Omega, \]

with \( m(x) > 0 \) for all \( x \in \Omega \). Hence according to Theorem 2.1., every positive weak solution of (18) is stable.

**Example 3.3.** Consider the singular chemotaxis steady-state problem of degree two,

\[-\text{div}(|x|^{-a} \nabla u) + p \nabla u = \lambda m(x)|x|^{-(a+1)p+c} [-u + u^2] \text{ in } \Omega, \]

\[Bu = 0 \text{ on } \partial \Omega, \]

with \( m(x) > 0 \) for all \( x \in \Omega \). As a sequence of Theorem 2.2., every positive weak solution of (19) is unstable.

### REFERENCES


