Journal of Mathematical Sciences and Optimization. e-ISSN:3025-3926
Vol. 2, No. 2 (2025), pp.135-143

A Sixth-Order Predictor-Corrector Method for Initial Value Problems

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Abstract
This article discusses the sixth-order predictor-corrector method by changing the integral limit of $[t_n, t_{n+1}]$ to $[t_{n-3}, t_{n+1}]$. This method combines the explicit Adam-Bashforth approach as a predictor and the implicit Adam-Moulton approach as corrector. The results obtained show that the numerical solution is close to the exact solution and the selection of a small stepsize h makes this method an alternative method in solving various initial value problems.

Keywords : *Initial value problems, predictor-corrector method, Adam-Bashforth method, Adam-Moulton method, stability region*

1. INTRODUCTION

Differential equations form a fundamental branch of applied mathematics, playing a crucial role in modeling natural phenomena and systems involving dynamic changes. As a subject, differential equations are strategically important because they integrate key aspects of mathematics-such as analysis, algebra, and geometry-which are essential for introducing core concepts and solving real-world problems.

One common issue in the study of differential equations is the initial value problem (IVP). The general form of an IVP is given by:

$$X'(t) = f(t, X(t)), \quad X(t_0) = X_0, \quad t \in \mathbb{R}, \quad t \geq t_0. \tag{1}$$

Equations (1) can be solved either analytically or numerically. However, the analytical solution is often difficult or impossible to obtain, making numerical methods a more practical and optimal approach. Numerical techniques for solving IVPs can be broadly classified into one-step methods, such as Euler and Runge-Kutta, and multi-step methods.

In one-step methods, Runge-Kutta (RK) serves as an alternative to Taylor methods, as it does not require evaluating derivatives of the function. RK methods are known for their high accuracy, although the number of computations required increases significantly with the order of the method.

As explained by Boyce and DiPrima [4], in multi-step methods, the integral in Equation (1) is computed over several intervals, allowing the use of previously computed solution points to generate more accurate approximations. This class includes Adams-Bashforth (AB) methods, which are explicit and computationally efficient but less stable, and Adams-Moulton (AM) methods, which are implicit and offer better stability at the cost of additional computations at each step.

Received: 05-04-2025, accepted: 07-05-2025

Atkinson and Han [1] describe multi-step methods as ebing derived by integrating Equation (1) over the interval $[t_n, t_{n+1}]$, leading to the following expression:

$$X_{n+1} = X_n + \int_{t_n}^{t_{n+1}} f(t, X(t))dt.$$

Building on this foundation, several researchers have explored the development and application of predictor-corrector methods. For example, Sami et al. [7] employed high-order predictor-corrector schemes to solve fractional-order differential equations, as did Binh and Bongsoo [3]. Zaid [8] also utilized a predictor-corrector method for solving initial value problems involving the Caputo fractional derivative.

In this article, we investigate the initial value problem given in Equation (1) using a predictor-corrector approach by extending the integration interval from $[t_n, t_{n+1}]$ to $[t_{n-4}, t_{n+1}]$. The study also includes a numerical stability analysis of the Adam-Bashforth (AB) and Adams-Moulton (AM) methods. Finally, the effectiveness of the proposed method is assessed through a series of numerical experiments on selected example problems. .

2. RESEARCH METHOD

2.1. Derivation of the Proposed Method

This study begins by defining the initial value problem in its general form. The differential equation is then integrated over the interval $[t_{n-4}, t_{n+1}]$, under the assumption that the solution values at the previous points are known. To determine the required weights or integral coefficients in the resulting formula, the Lagrange polynomial interpolation approach is employed. This technique enables a more efficient estimation of the integral. The resulting coefficients are then used to construct a sixth-order predictor formula, which provides a more accurate initial numerical approximation before correction.

2.2. Method Development

In this phase, the derived predictor method is further developed by introducing a corrector method, leading to the construction of a sixth-order predictor-corrector scheme. The development begins by using the fourth-order Runge-Kutta (RK-4) method to generate the initial values required for the multi-step process.

Once the initial values are obtained, the sixth-order Adams-Bashforth method (AB-6) is used as a predictor to estimate the solution at the next time step. This prediction yields an approximation of x_{n+1} without requiring iterations or the solution of nonlinear equations. As an explicit method, AB-6 offers fast computation but tends to produce higher error. Therefore, the predicted value is refined using a corrector method. The corrector employed is the sixth-order Adams-Moulton method (AM-6), which is implicit and delivers higher accuracy by incorporating the newly estimated solution value.

The corrector method is derived similarly to the predictor, using integral approximation, with coefficients determined through Lagrange polynomial interpolation. These coefficients are then substituted into the integral expression, resulting in the formulation of the corrector.

3. RESULTS

3.1. Fourth-Order Runge-Kutta Method

The Runge-Kutta method serves as an alternative to the Taylor series method, offering a way to approximate solutions without requiring derivative computations. As described by Burden and Faires [5], the Runge-Kutta method evaluates $f(t, X(t))$ at several points within each step, while maintaining an accuracy level comparable to that of the Taylor method. Within the predictor-corrector framework, the Runge-Kutta method is typically used to generate the initial estimates required before the multi-step predictor-corrector scheme can be applied.

Among the various Runge-Kutta methods, the fourth-order Runge-Kutta (RK-4) method is the most widely used due to its superior accuracy compared to lower-order methods. The general form of the fourth-order Runge-Kutta method is given as follows:

$$x_{n+1} = x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

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$$\begin{aligned} k_1 &= f(t_n, x_n) \\ k_2 &= f(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_1) \\ k_3 &= f(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_2) \\ k_4 &= f(t_n + h, x_n + hk_3). \end{aligned}$$

3.2. Adam-Bashforth-Moulton Predictor Method

The Adams-Bashforth method is an explicit technique used to solve initial value problems of the form given in Equation (1). As explained by Boyce and DiPrima [4], this method acts as a predictor, utilizing previously computed solution values to estimate the derivative function. The sixth-order Adams-Bashforth method (AB-6) is derived by integrating Equation (1) over the interval $[t_{n-3}, t_{n+1}]$,

$$X_{n+1} = X_{n-3} + \int_{t_{n-3}}^{t_{n+1}} g(t)dt, \tag{2}$$

where $g(t) = f(t, X(t))$ is approximated with Lagrange interpolating polynomial of order sixth, $g(t) \approx P_5(t)$

$$g(t) = P_5(t) + R_5(t), \tag{3}$$

where $P_5(t)$ is a Lagrange interpolating polynomial, and L is a Lagrange polynomial, which is defined as follows:

$$\begin{aligned} P_5(t) = {} & g(t_{n-5})L_{n-5}(t) + g(t_{n-4})L_{n-4}(t) + g(t_{n-3})L_{n-3}(t) + g(t_{n-2})L_{n-2}(t) \\ & + g(t_{n-1})L_{n-1}(t) + g(t_n)L_n(t), \end{aligned}$$

where $R_5(t)$ is the interpolation error defined as follows:

$$R_5(t) = \frac{(t - t_{n-5})(t - t_{n-4})(t - t_{n-3})(t - t_{n-2})(t - t_{n-1})(t - t_n)}{(6)!} f^{(6)}(\xi_t).$$

Based on the previously described integration process, the resulting for Equation (2) is given as follows:

$$\begin{aligned} X_{n+1} = X_{n-3} + \frac{h}{45}(& 148f(t_n, X_n) - 186f(t_{n-1}, X_{n-1}) + 344f(t_{n-2}, X_{n-2}) - 196f(t_{n-3}, X_{n-3}) \\ & + 84f(t_{n-4}, X_{n-4}) - 14f(t_{n-5}, X_{n-5})) + \frac{286}{945}h^7 X^{(6)}(\xi_t). \end{aligned} \tag{4}$$

Let $x_n = x_{(t_n)} \approx X_{(t_n)}$, where x_n is the approximation to the exact solution $X(t_n)$. From Equation (4), the *AB-6* is given as follows:

$$x_{n+1} = x_{n-3} + \frac{h}{45}(148x'_n - 186x'_{n-1} + 344x'_{n-2} - 196x'_{n-3} + 84x'_{n-4} - 14x'_{n-5}), \tag{5}$$

where $n = 3, 4, 5, \ldots$.

From (5), we obtain the sixth order predictor method as follows:

$$x^{(0)}_{n+1} = x_{n-3} + \frac{h}{45}(148f_n - 186f_{n-1} + 344f_{n-2} - 196f_{n-3} + 84f_{n-4} - 14f_{n-5}). \tag{6}$$

3.3. Adam-Bashforth-Moulton Corrector Method

The corrector method is derived using the same process as the predictor method. In the corrector step, f_{n+1} is evaluated using the sixth-order Adams-Moulton method to obtain a more accurate value of x_{n+1}. Based on Equation (3), the formulation for the sixth-order Adams-Moulton method is given by:

$$P_5(t) = g(t_{n-4})L_{n-4}(t) + g(t_{n-3})L_{n-3}(t) + g(t_{n-2})L_{n-2}(t) + g(t_{n-1})L_{n-1}(t) + g(t_n)L_n(t) + g(t_{n+1})L_{n+1}(t), \tag{7}$$

where $P_5(t)$ is the Lagrange interpolating polinomial and the interpolation error $R_5(t)$ is given by

$$R_5(t) = \frac{(t-t_{n-4})(t-t_{n-3})(t-t_{n-2})(t-t_{n-1})(t-t_n)(t-t_{n+1})}{(6)!} f^{(6)}(\xi_t). \tag{8}$$

Integrating Equation (7) and (8), and substituting to (2) resulting in:

$$X_{n+1} = X_{n-3} + \frac{h}{45}(14f(t_{n+1},X_{n+1}) + 64f(t_n,X_n) + 24f(t_{n-1},X_{n-1}) + 64f(t_{n-2},X_{n-2}) + 14f(t_{n-3},X_{n-3})) - \frac{8}{945}h^7 X^{(6)}(\xi_t). \tag{9}$$

By letting $x_n = x_{(t_n)} \approx X_{(t_n)}$ in (9), we obtain *AM-6* as described below:

$$x_{n+1} = x_{n-3} + \frac{h}{45}(14x'_{n+1} + 64x'_n + 24x'_{n-1} + 64x'_{n-2} + 14x'_{n-3}), \tag{10}$$

where $n = 2,3,4,\ldots$.

We define formula for *AM-6* as:

$$x_{n+1} = x_{n-3} + \frac{h}{45}(14f_{n+1} + 64f_n + 24f_{n-1} + 64f_{n-2} + 14f_{n-3}). \tag{11}$$

3.4. Stabiliy of Adam-Bashforth-Moulton Method

Atkinson and Han [1] state that a numerical method is considered stable if the chosen step size h is sufficiently small. However, selecting a very small h may reduce computational efficiency. Stability refers to the ability of a numerical method to maintain a bounded and accurate solution, even when the step size is relatively large.

The Adams-Bashforth method, due to its explicit nature, has a relatively narrow stability region. As a result, the step size h must be kept small to prevent instability in the numerical solution. Atkison [2] provide a commonly used test problem for analyzing stability:

$$X'(t) = \lambda X(t), \quad t > 0, \quad X(0) = 1, \tag{12}$$

where λ is a real negative number or a complex number with a negative real part. The exact solution to Equation (12) is $X(t) = e^{\lambda t}$.

A numerical method is said to be stable for this problem if its solution satisfies $x(t_n) \to 0$ as $t_n \to \infty$ regardless the chosen step size h. The set of all $h\lambda$ values in the complex plane for which the numerical solution satisfies $x_n \to 0$ as $n \to \infty$ is known as the method's region of absolute stability [2].

Applying the *AB-6* for the test problem (2) in order to determine the stability of the method, results in

$$x_{n+1} = x_{n-3} + \frac{h\lambda}{45}(148x_n - 186x_{n-1} + 344x_{n-2} - 196x_{n-3} + 84x_{n-4} - 14x_{n-5}), \tag{13}$$

where $n = 3,4,5,\ldots$. Next, by letting $x_n = r^n$ in equation (13) and utilizing the characteristics polynomial, we have

$$r^{n+1} - r^{n-3} - \frac{h\lambda}{45}(148x_n - 186x_{n-1} + 344x_{n-2} - 196x_{n-3} + 84x_{n-4} - 14x_{n-5}) = 0, \tag{14}$$

where $n = 3,4,5,\ldots, r = e^{i\theta}$, and $0 \le \theta \le 2\pi$. Equation (14) is used to solve the characteristic roots that will be used to determine the stability of the method.

Furthermore, by simplifying equation (14) dan by letting $h\lambda = z$, we obtain the stability polynomial for the sixth-order Adam-Bashforth method as follows:

$$z = \frac{45(e^{5i\theta} - e^{i\theta})}{148e^{5i\theta} - 186e^{4i\theta} + 344e^{3i\theta} - 196e^{2i\theta} + 84e^{i\theta} - 14}. \tag{15}$$

If the polynomial in Equation (15) is plotted for $0 \le \theta \le 2\pi$ the stability region of the Adams-Bashforth sixth-order (*AB-6*) method is obtained, as illustrated in Figure 1 below. The area enclosed within the curve represents the stability region of the *AB-6* method.

As shown in the figure, this stability region includes a portion of the negative real axis up to a certain bound, indicating that the method is particularly suitable for solving differential equations whose stability can be maintained by appropriately choosing the step size h. ‘

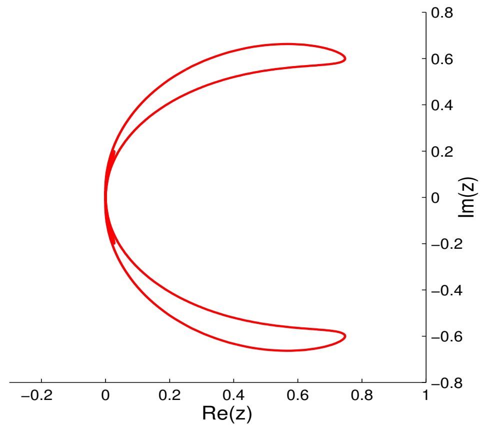

FIGURE 1. The stability region of the sixth-order Adam-Bashforth method

After obtaining the stability region of the *AB-6* method, the stability region of the *AM-6* method will be determined using the same procedure. By applying the test problem from Equation (2), the *AM-6* method formula in Equation (10) becomes:

$$x_{n+1} = x_{n-3} + \frac{h\lambda}{45}(14x_{n+1} + 64x_n + 24x_{n-1} + 64x_{n-2} + 14x_{n-3}), \tag{16}$$

where $n = 2, 3, 4, \ldots$. Next, by using the characteristics polynomial approach and by letting $x_n = r^n$, Equation (16) becomes

$$r^{n+1} - r^{n-3} - \frac{h\lambda}{45}(14x_{n+1} + 64x_n + 24x_{n-1} + 64x_{n-2} + 14x_{n-3}) = 0, \tag{17}$$

where $n = 3, 4, 5, \ldots, r = e^{i\theta}$ dan $0 \le \theta \le 2\pi$.

By simplifying Equation (17) and by assuming $h\lambda = z$, we obtain stability polynomial for the sixth-order AM method as follows:

$$z = \frac{45(e^{4i\theta} - 1)}{14e^{5i\theta} + 64e^{4i\theta} + 24e^{3i\theta} + 64e^{2i\theta} + 14e^{i\theta}}. \tag{18}$$

If the polynomial in Equation (18) is plotted for $0 \le \theta \le 2\pi$ the stability region of the *AM-6* method is obtained, as illustrated in Figure 2 below. The area enclosed by the curve represents the stability region of the *AM-6* method. Unlike the Adams-Bashforth method, the Adams-Moulton method exhibits a wider stability region, making it more suitable for solving differential equations where stability can be preserved by selecting an appropriate step size h. The stability region of the *AM-6* method indicates that, at certain orders, the method possesses strong stability properties. It is particularly effective for numerical simulations that demand high stability, and is therefore frequently used in computations requiring high accuracy and long-term integration.

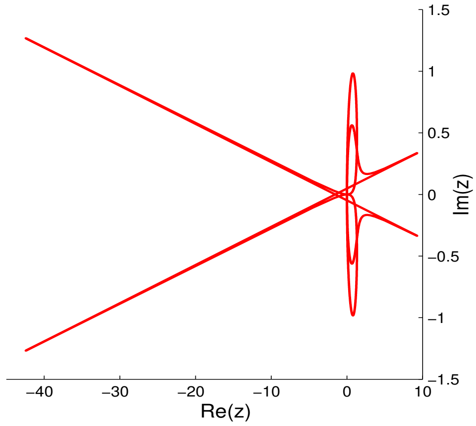

FIGURE 2. The stability region of the sixth-order Adam-Moulton method

3.5. Numerical Simulation

In this section, we applied the proposed method to several initial value problems as follow:

Example 3.1. *The solution of the initial value problems*

$$X'(t) = (2-t)x(t), \quad 2 \le t \le 3, \quad x(2) = 1, \tag{19}$$

with the exact solution

$$X(t) = e^{-0.5(t-2)^2}.$$

The solution of (19) in Example 3.1 using the sixth-order predictor-corrector method is presented in Table below, where x_n denotes the numerical approximation of x_{n+1}, and X_n represents the exact solution.

TABLE 1. Solution for Example 3.1.

t_n	x_n	E	E_{rel}
2.00	1.0000	0	0
2.50	0.8825	$1.41e-12$	$1.60e-12$
3.00	0.6065	$2.79e-12$	$4.61e-12$
3.50	0.3247	$1.04e-12$	$3.22e-12$
4.00	0.1353	$6.07e-13$	$4.49e-12$
4.50	0.0439	$4.10e-13$	$9.34e-12$
5.00	0.0111	$2.69e-13$	$2.42e-11$
5.50	0.0022	$9.09e-14$	$4.16e-11$
6.00	0.0003	$1.80e-13$	$5.37e-10$

In Table 1, the error is defined as the absolute difference between the exact solution X_n and the numerical solution x_n, expressed as $E = |X_n - x_n|$, and the relative error is given by $E_{rel} = |\frac{X_n - x_n}{X_n}|$. The method is implemented with a step size of $h = 0.02$ over the interval $t = [2, 6]$.

If the solution from Example 3.1 is plotted, the resulting graph is illustrated in Figure below. As shown in the figure, the numerical solution closely follows the exact solution, indicating good agreement between the two. The plot also reveals that the most significant error occurs at the beginning of the computation. However, the error decreases over time, suggesting that the accuracy of the numerical method improves as the solution progresses.

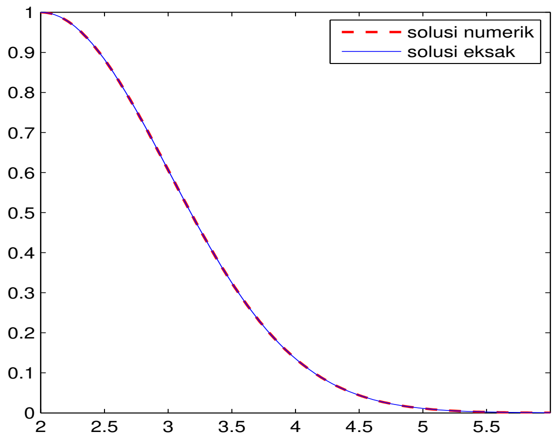

FIGURE 3. Graph of the solution Example 3.1 with the sixth-order predictor-corrector method

Example 3.2. [6] *Water balance observations in peatland areas under daily rainfall are carried out through simulations based on available data. The governing equation for the water balance is given by:*

$$\frac{dV}{dt} = A(P - ET) - Q. \tag{20}$$

where $V(t)$ represents the water volume at time t, $A = 1000m^2$ is the observation area, $ET = 5mm/day$ is the constant evapotranspiration rate, and $Q = 20m^3/day$ is the estimated outflow due to surface runoff and drainage. THe rainfall function is modeled as $P(t) = 10 + 5\ sin\ (0.01\pi t)$, which follows as sinusoidal pattern representing periodic weather changes over a span of 100 days. The goal is to predict the water volume over a $100-day$ period, starting from an initial volume of $V(0) = 2000m^3$, using the predictor-corrector method. The problem setup and its simulation results are illustrated in figure below.

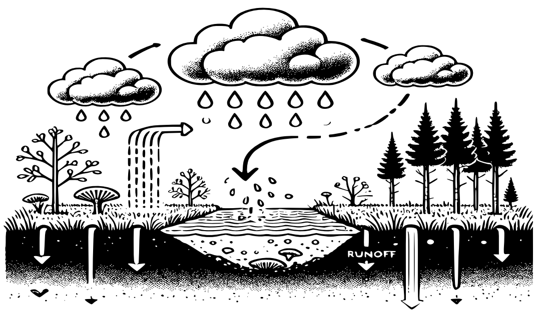

FIGURE 4. IIllustration for Example 3.2

The solution for Example 3.2 using the sixth-order predictor-corrector method is presented in Table 2, where V denotes the computed water volume at time t_{n+1} and X_n represents the exact (analytical) solution at the corresponding time step. In the following table, the error is calculated as the absolute difference between the exact and numerical solutions, expressed as$E = |X_n - V|$, while the relative error is given by $E_{rel} = |\frac{X_n - V}{X_n}|$. The simulation is performed over the interval $t = [0, 100]$ with a step size of $h = 1$.

The solution of Example 3.2 is illustrated in Figure 5, which depicts the variation of water volume $V(t)$ in a peatland area over a 100-day period, based on the water balance model described

TABLE 2. Solution for Example 3.2.

t_n	$V(m^3)$	E	E_{rel}
0	2000	0	0
10.00	1857.7896	$9.12e-11$	$4.91e-14$
20.00	1730.3959	$3.65e-10$	$2.11e-13$
30.00	1615.6060	$2.66e-11$	$1.65e-14$
40.00	1509.9734	$2.03e-10$	$1.35e-13$
50.00	1409.1549	$2.17e-10$	$1.54e-13$
60.00	1308.3365	$3.18e-12$	$2.43e-15$
70.00	1202.7039	$4.07e-10$	$3.38e-13$
80.00	1087.9140	$1.59e-10$	$1.46e-13$
90.00	960.5203	$5.25e-10$	$5.46e-13$
100.00	818.3099	$2.21e-10$	$2.70e-13$

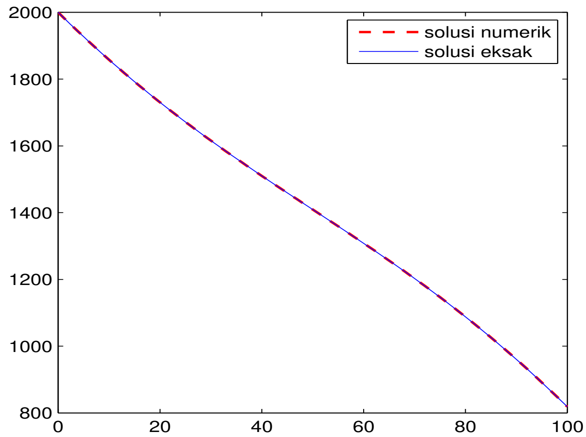

FIGURE 5. Graph of the solution for Example 3.2

by Equation (20). The figure clearly shows that the numerical and exact solutions closely overlap, indicating that the predictor-corrector method yields highly accurate results.

The observed decline in water volume is primarily attributed to the sinusoidal pattern of rainfall $P(t)$, which periodically fluctuates and is often insufficient to offset the combined effects of constant evapotranspiration ET and outflow Q. Consequently, the water volume in the system continuously decreases over time. This declining trend reflects the long-term water balance and provides valuable insight into how peatland ecosystems respond to climatic variations and rainfall patterns. The close agreement between the numerical and exact solutions demonstrates the effectiveness and reliability of the predictor-corrector method in solving dynamic models of this nature.

4. CONCLUSIONS

Based on the results and discussions presented in this study, the numerical experiments demonstrate that the predictor-corrector method yields numerical solutions that closely approximate the exact solutions. The discrepancy between the numerical and exact solutions was evaluated using the maximum norm $\|X(t_n)-x(t_n)\|$, which effectively quantifies the accuracy of the numerical method.

Furthermore, the tabulated results indicate a clear trend: as the stepsize h decreases, the numerical error also diminishes. This highlights the sensitivity of the method to the choice of stepsize and confirms that optimal accuracy can be achieved with appropriately small stepsizes. Thus, the performance of the predictor-corrector method, particularly of sixth order, depends significantly on a well-chosen discretization parameter. These findings underscore the method's reliability and effectiveness for solving initial value problems with high precision, especially when computational resources allow for fine time discretization.

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