Journal of Mathematical Sciences and Optimization
Vol. 1. No. 1 (2023), pp. 17-24

Optimal Inventory Policy Using Assignment Model As Mixed Integer Linear Programming Problem

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Abstract
This article discusses the assignment model as a mixed integer linear programming problem used in determining inventory policy for the multi-item case with backorder using the branch-and-bound method. The assignment model used must meet the specified constraints. The LINGO 18.0 application is used to solve mixed integer linear programming problems from the inventory problem assignment model. The optimal solution obtained from the assignment model uses the LINGO 18.0 application, which is based on the branch-and-bound method compared to the EOQ model with backorder. The results obtained from the comparison of these two models can be concluded to show that the inventory problem provides more optimal value using the assignment model as a mixed integer linear programming problem compared to using the EOQ model with backorder.

Keywords: *Inventory problems, backorder, assignment models, mixed integer linear programming problems*

1. INTRODUCTION

The growing economy encourages global market competition in technology and information to determine optimal resources for a company. An optimal inventory policy will reduce a company's inability to meet customer demands on time [4].

Appropriate mathematical modeling in describing the inventory system of an inventory problem is needed in order to produce an optimal policy. The EOQ model with backorder is a model that is often used to describe inventory systems with inventory problems [6]. As time went by, experts began to research inventory models, including Begnaud et al. [1], who used the notion of echelon cut in a branch-and-cut algorithm to formulate a multi-level lot size inventory problem from a mixed integer linear programming of inventory problems. Furthermore, Surati [7] determines production decisions in order to produce minimum costs in the inventory system by developing a mixed integer programming model. Khalil et al. [4] determine the optimal inventory policy using an assignment model to describe an inventory system that is formulated as a mixed integer linear programming problem on an inventory problem with backorders. The structure of this article includes a second part explaining the EOQ model with backorder. The third part is a mixed integer linear programming problem and the branch-and-bound method. The fourth section discusses the assignment model as a mixed-integer linear programming problem. Then the fifth part compares the results obtained from solving an inventory problem with the proposed model and the EOQ model with backorders to obtain the optimal inventory policy.

2. EOQ MODEL WITH BACKORDER

The EOQ model with backorder is a model that allows for a vacancy of goods so that the order request is not fulfilled on time and causes a backorder. Order fulfillment is promised to occur in the next period [2]. The assumptions used in this model include the following [8]:

(i) Demand is known with certainty and is constant.
(ii) The lead time for the receipt of orders is constant.
(iii) The order quantity is received all at once.
(iv) The ordering cost per order is fixed.

(v) Holding costs and backorder costs are set per unit, per order, and per unit time.
(vi) Purchase costs are constant throughout the planning period.
(vii) Backorder is allowed or permitted.

In the EOQ model with backorder inventory parameters used are ordering costs, purchasing costs, holding costs and backorder costs. The combination of the sum of the inventory parameters used will produce the total inventory cost, which can be written as follows [10]:

$$TC_b = pD + \frac{q^2 h}{2Q} + \frac{sD}{Q} + \frac{(Q-q)^2 b}{2Q}.$$

Where:

Q	≔	Economic inventory demand quantity per order,
p	≔	Purchasing cost of inventory item per unit of time,
q	≔	Maximum inventory level per unit of time,
D	≔	Inventory demand per unit of time,
s	≔	Ordering cost of inventory each time an order is placed,
h	≔	Holding cost of inventory item per unit of time,
b	≔	Backorder costs of item per unit of time.

The next step is determining the first derivative of TC_b to Q and q which must be equal to 0 as follows [11]:

$$\frac{\partial TC_b}{\partial Q} = \frac{\partial}{\partial Q}\left(pD + \frac{q^2 h}{2Q} + \frac{sD}{Q} + \frac{(Q-q)^2 b}{2Q}\right) = 0, \tag{1}$$

and

$$\frac{\partial TC_b}{\partial q} = \frac{\partial}{\partial q}\left(pD + \frac{q^2 h}{2Q} + \frac{sD}{Q} + \frac{(Q-q)^2 b}{2Q}\right) = 0. \tag{2}$$

From equation $(1) - (2)$ obtained Q^* and q^*, that is

$$Q^* = \sqrt{\frac{2sD}{h}}\sqrt{\frac{h+b}{b}} \text{ dan } q^* = \sqrt{\frac{2sD}{h}}\sqrt{\frac{h}{h+b}}.$$

Furthermore, determining the second derivative of TC_b with a Hessian matrix of the form

$$\nabla^2 TC_b(Q,q) = \begin{bmatrix} \frac{\partial^2 TC_b}{\partial^2 QQ} & \frac{\partial^2 TC_b}{\partial^2 Qq} \\ \frac{\partial^2 TC_b}{\partial^2 qQ} & \frac{\partial^2 TC_b}{\partial^2 qq} \end{bmatrix},$$

TC_b is obtained where $Q^*$and q^* produce a positive definite which proves that

$$TC_b = \frac{{q^*}^2 h}{2Q^*} + \frac{sD}{Q^*} + \frac{(Q^* - q^*)^2 b}{2Q^*},$$

is a minimum total inventory cost in the EOQ model with backorder. In the case of backorder inventory, shortages that occur can be seen through the following equation [11]:

$$q_s = Q - q$$

Furthermore, the frequency of orders and the time between inventory orders in the EOQ model with backorders are obtained through the following equation [9]:

$$\text{Order frequency} = \frac{\text{Inventory demand per unit of time}}{\text{Inventory demand quantity per order}}$$

$$F = \frac{D}{Q}$$

$$\text{Times between of orders} = \frac{\text{Inventory demand per unit of time}}{\text{Inventory demand quantity per order}},$$

$$W_p = \frac{W}{F}.$$

3. MIXED INTEGER LINEAR PROGRAMMING PROBLEM AND BRANCH AND BOUNCH METHOD

Integer linear programming problems appear when we have to decide on the number of items needed in the form of integers, provided that some or all of the variables are limited to integers or discrete values. In general, the form of a mixed integer linear programming problem is as follows: [4]:

$$\max z = \sum_{j=1}^{n} c_j x_j,$$

subject to

$$\sum_{j=1}^{n} a_{ij}\, x_j \le b_i \,;\, i = 1,2,\dots,m,$$

$$x_j \ge 0 \text{ ; integer and } n \le m \text{ or } x_j = 0 \text{ or } 1,\ j = 1,2,\dots,n,$$

where:

- z ≔ scalar value of decision criteria or objective function,
- x_j ≔ integer linear programming decision variables, $j = 1, 2, \dots, n$,
- c_j ≔ the coefficient of the decision variable in the objective function , $j = 1, 2, \dots, n$,
- b_i ≔ constraint function value, $i = 1, 2, \dots, m$,
- a_{ij} ≔ constraint decision variable coefficient, $i = 1, 2, \dots, m,\ j = 1, 2, \dots, n$,
- n ≔ number of decision variables,
- m ≔ number of constrain.

The basic method that can be used to solve integer linear programming problems is the branch-and-bound method, with the steps for each iteration as follows [4]:

(i) Setting $z^* = \infty$, then determining the linear programming relaxation of the mixed integer linear programming problem.

(ii) Applying the simplex method to the linear relaxation program so that the optimal solution for the linear relaxation program is obtained.

(iii) Considering the optimal solution obtained using the simplex method, if the decision variable obtained is an integer, the integer optimum has been reached. If one or more of the variables obtained are not integers, proceed to the next step.

The steps for each subsequent iteration are as follows:

(a) *Branching*. Among all remaining (unfathomed) subproblems, select the one that was created recently and has the greatest value. Form a branch from a subproblem point to create two new subproblems. Suppose x^* and x_j are values in the solution. Branch this point by creating a new subproblem by adding the constraint functions in sequences $x_j \leq \lfloor x_j^* \rfloor$ and $x_j \geq \lfloor x_j^* \rfloor + 1$. Meanwhile, for the zero-one integer programming problem, the variables assigned to each other are 1 and 0, respectively.
(b) *Bounding*. For each new subproblem, find the constraints by applying the simplex method to each relaxation of the linear program and floor on the z^* value for the resulting optimal solution.
(c) *Fathoming*. For each new subproblem, apply the three fathomed tests and discard the subproblems that were disconnected from the tests performed as follows:
    (i) The limit value is less than or equal to ($\leq$) z^* for the maximum objective function.
    (ii) The obtained linear relaxation program solution is not feasible.
    (iii) The optimal solution for a linear relaxation program with integer values (if the solution is more optimal than the previous limit, this solution becomes a new limit, and the test is applied again so that the value is obtained z^* more optimal).
(d) Optimality test. The calculation is stopped if there are no further subproblems with the incumbent obtained is optimal.

4. ASSIGNMENT MODEL AS MIXED INTEGER LINEAR PROGRAMMING PROBLEM IN INVENTORY PROBLEM

The assignment model used to determine the optimal inventory policy is built based on the optimal assignment technique. The assumption of the assignment model as a mixed integer linear programming problem is as follows [5]:
(i) Demand is known and varies with time.
(ii) The purchase cost per unit in the planned period is constant.
(iii) Fulfillment of requests in the planned period can be obtained at any time, including backorder.
(iv) The lead time is known with certainty.
(v) No discounted quantity is permitted.
(vi) Holding costs and backorder costs are zero if the total quantity demanded is obtained in the same period.

The assignment model as a mixed integer linear programming problem for multi-item inventory problems can be arranged in a cost table that describes i delivery periods and j inventory received periods for n items, which can be seen in Table 1 as follows:

Table 1: Assignment Model Inventory System

Item No. n	Period				
Period	1	2	3	$\cdots$	m
1	0	h_{12}^n	h_{13}^n	$\cdots$	h_{1m}^n
2	b_{21}^n	0	h_{23}^n	$\cdots$	h_{2m}^n
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	b_{m1}^n	b_{m2}^n	b_{m3}^n	$\cdots$	0

The mathematical model of the assignment model as a mixed integer linear programming problem for the backorder inventory problem can be written as follows:

$$min\ z = \sum_{i=1}^{m} \sum_{j=i+1}^{m} \sum_{k=1}^{n} h_{ijk}\, x_{ijk} + \sum_{i=2}^{m} \sum_{j=1}^{i-1} \sum_{k=1}^{n} b_{ijk}\, x_{ijk} + \sum_{i=1}^{m} S_i y_i,$$

subject to

$$\sum_{j=1}^{m} x_{ijk} \leq Q_k y_i \; ; \; i = 1, 2, \dots, m \; ; k = 1, 2, \dots, n,$$

$$\sum_{i=1}^{m} x_{ijk} \leq d_{jk} \; ; \; j = 1, 2, \dots, m \; ; k = 1, 2, \dots, n,$$

$$y_i = 0, 1 \; ; i = 1,2, \dots, m,$$

$$x_{ijk} \geq 0, \; i = 1, 2, \dots, m \; ; j = 1, 2, \dots, m \; ; k = 1, 2, \dots, n,$$

where:

m := The number of periods in the planning horizon,

n := Number of items ordered for inventory,

x_{ijk} := The amount of inventory acquired in period i for demand in period j, for item k,

h_{ijk} := Unit holding costs for inventory in period i and received in period j, for item k,

b_{ijk} := Backorder cost per unit for inventory in period i and received in another horizon period j, for item k,

d_{jk} := The number of units needed for each item in period j for item k,

S_i := Set up cost of inventory in period i,

Q_k := The number of quantity units of each item k,

y_i := $\begin{cases} 1, & \text{if there is an order in periode } i \\ 0, & \text{if no there is an order in period } i. \end{cases}$

5. APPLICATION OF THE MODEL AND THE SOLUTION

Example 1 A company has placed an order with a regular supplier for several years for a variety of stationery products it sells. The company has an order period of one year. The company places orders for a period of six periods in one year with the average quantity required of various types of stationery products as can be seen in Table 2.

Table 2: Stationery products needed each period (units)

Item	Period 1	Period 2	Period 3	Period 4	Period 5	Period 6
Book	200	100	100	200	200	100
Pen	480	560	425	460	560	400
Pencil	320	200	275	280	245	320
W. Marker	240	110	160	230	210	150
Ballpoint	80	70	75	85	75	70
Remover	310	390	410	350	410	370
Note	145	100	140	250	195	130
Folder	190	130	195	170	110	185

A shortfall in inventory of products to be sold to the company causes the emergence of backorder costs. In addition, each time an order is placed, the company has to pay the ordering cost. Ordering stationery in lots causes inventory holding costs to appear. Backorder costs, ordering costs, and holding costs of the company can be seen in Tables 3, 4, and 5.

Table 3: Stationery product storage costs per unit

Item	Period 1	Period 2	Period 3	Period 4	Period 5
Book	40	80	120	160	200
Pen	52	108	156	208	260

Pencil	55	110	165	220	275
White marker	35	70	105	140	175
Ballpoint	27	54	81	108	135
Remover	33	66	99	132	165
Note	15	30	45	60	75
Folder	38	76	114	152	190

Table 4. Stationery product backorder cost per unit

Item	Period 1	Period 2	Period 3	Period 4	Period 5
Book	76	90	105	121	138
Pen	82	98	115	132	151
Pencil	87	104	122	141	161
W.Marker	63	75	88	101	116
Ballpoint	41	50	61	73	86
Remover	58	69	81	95	110
Note	32	39	47	56	67
Folder	74	88	102	117	133

Tabel 5: Ordering cost

	Ordering cost (Rp)
Period 1	80000
Period 2	80000
Period 3	80000
Period 4	80000
Period 5	80000
Period 6	80000

The assignment model as a mixed integer linear programming problem for the stationery company inventory problem is as follows:

$$\min z = \sum_{i=1}^{6} \sum_{j=i+1}^{6} \sum_{k=1}^{8} h_{ijk}\, x_{ijk} + \sum_{i=2}^{6} \sum_{j=1}^{i-1} \sum_{k=1}^{8} b_{ijk}\, x_{ijk} + \sum_{i=1}^{6} S_i y_i,$$

subject to

$$\sum_{j=1}^{6} x_{ijk} \leq Q_k y_i \ ;\ i = 1, 2, \dots, 6 \ ;\ k = 1, 2, \dots, 8,$$

$$\sum_{i=1}^{m} x_{ijk} \leq d_{jk} \ ;\ j = 1, 2, \dots, 6 \ ;\ k = 1, 2, \dots, 8,$$

$$y_i = 0, 1 \ ;\ i = 1,2, \dots, 6,$$

$$x_{ijk} \geq 0,\ i = 1, 2, \dots, 6 \ ;\ j = 1, 2, \dots, 6 \ ;\ k = 1, 2, \dots, 8.$$

By solving the company's inventory problem with the assignment model as a mixed integer linear programming problem using the branch-and-bound method with LINGO 18.0, it was found that the company had to order inventory three times, that was, in the first period, third period, and fifth period with one period's ordering cycle time was four months. Orders of the first period to cover the amount of inventory needed for the first and second periods. Third period orders to cover the amount of inventory needed for the third and fourth periods. Ordering the fifth period to cover the inventory needs for the fifth and sixth periods. The number of units in the company's inventory orders for each item according to the order of purchase can be seen in Figure 1.

Furthermore, solving the same inventory problem using the EOQ model with backorders. The results of calculating the number of economical orders, maximum inventory levels, maximum inventory shortage levels, number of ordering frequencies, time between orders, and total inventory costs for each company inventory item using the EOQ model with backorders can be summarized in Table 6.

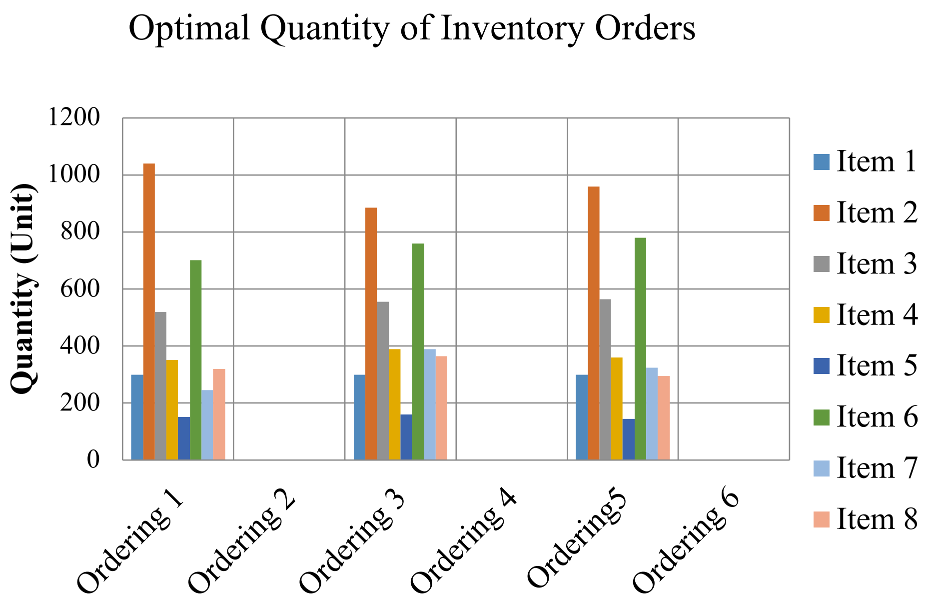

Figure 1: The order quantity of each stationery company item
The company's total inventory cost in one year based on LINGO 18.0 is Rp459,325

Table 6: Company Inventory Problem Calculation Results

	Q	q	q_s	F	W_p	$TC_b(Rp)$
Item 1	1328	542	786	1	12	108.437
Item 2	2198	808	1391	1	12	209.984
Item 3	1607	594	1014	1	12	163.237
Item 4	1588	633	955	1	12	110.805
Item 5	1177	458	719	1	12	61.842
Item 6	2330	932	1398	1	12	153.800
Item 7	2083	983	1100	1	12	73.726
Item 8	1416	583	833	1	12	110.758
Total cost inventory						992.589

Based on Table 6, the total inventory cost of the company using the EOQ model with backorders in one year is Rp992,589. The results obtained for the company's inventory problem with the assignment model as a mixed integer linear programming problem and the EOQ model with backorder show that the assignment model as a mixed integer linear programming problem produces a minimum total cost. Inventory problem solving using the assignment model as a mixed integer linear programming problem can be an alternative to solving an inventory problem. The total cost of the resulting inventory is lower than using the EOQ model with backorders. Thus, the assignment model as a mixed integer linear programming problem can be used as an optimal inventory policy in comparison with the EOQ model with backorders by minimizing the number of inventory orders from inventory problems that happen at a company.

Acknowledgement. Authors appreciate the assistance of Prof. Dr. M. D. H. Gamal, M. Sc. throughout the process of writing this article.

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