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Edge Irregular Reflexive Labeling on Lobster Graph

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Abstract
Let G be a lobster graph that have three layer of vertices where each layer is connected to each other. The total labeling of the graph is called an edge irregular reflexive k-labeling if the total weight of two incident vertices and the edge that joins it is different for all possible edges on the graph. In this paper, we will further discuss the minimum number of k for this kind of labeling on lobster graph. In particular, we determine the exact value of the reflexive edge strength of lobster graph. To help illustrate it, we use Python code to generate the label for the graph.
Keywords : lobster graph, total labeling, irregular labeling, reflexive labeling, reflexive edge strength.

1. INTRODUCTION

Let G be a graph with a vertex set $V(G)$ and an edge set $E(G)$. The graph that we will defined here are finite, connected, undirected, and simple. In graph theory, vertex labeling are any mapping that maps a vertex set of graph elements to a set of numbers and edge labeling are any mapping that maps an edge set of graph elements to a set of numbers [4]. We call the labeling as total labeling if the domain is all of the vertex and edges.

One interesting graph to study is called a lobster graph. The term "lobster graph" refers to a particular class of tree-like graph that looks like a lobster. Lobster graphs are interesting structures in graph theory and have been studied in various contexts, including network analysis and optimization problems. They exhibit certain unique properties due to their tree-like structure with some pendent edges attached to the central path.

The notion of edge irregular reflexive labeling is described by Baca et al. [2]. Basically, an irregular labeling of a graph is an assignment of positive integers to the vertices or edges of the graph such that no two vertices (or edges) receive the same label, and the differences between the labels of adjacent vertices (or edges) are distinct. In other words, if two vertices (or edges) are adjacent, the absolute difference between their labels must be unique. Reflexive labeling is a special case of irregular labeling where vertices (or edges) are also allowed to have a label of 0. The inclusion of label 0 allows the possibility of having loops in the graph. Reflexive labeling is an extension of irregular labeling and can be applied to a broader class of graphs, including those with self-loops.

There are some recent results on the edge irregular reflexive labeling by Agustin et al. [5] on some tree graphs, Indriati et al. [6] on corona of path and other graphs, Tanna et al. [8] on prisms and wheels, and Zhang et al. [9] on the disjoint union of gear graphs and prism graphs. Many recent results also emerge on the total edge irregular reflexive strength by Baca and Siddiqui [3] on generalized prism, Indriati et al. [7] on generalized helm, and Ahmad [1] on the zigzag graphs. In this research, we will further discuss the edge irregular reflexive labeling on lobster graph. This lobster graph is unique in the sense there are many layer of vertices that are connected to each other to resemble the picture of a sea animal lobster.

In this paper, we determine the minimum span value of the edge irregular reflexive labeling for lobster graph. We use a theorem as a basis for finding the minimum range value of the reflexive edge strength. Next we propose an algorithm and then implement it later to Python for the automation of the construction of edge irregular reflexive labelling on lobster graph. Based on this algorithm, we can get and then prove the exact value of reflexive edge strength on lobster graph.

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2. PRELIMINARIES

Let G be a graph, denoted as $G = (V, E)$, where $V = \{v_1, v_2, ..., v_n\}$ is a non-empty set of vertices and $E \subseteq \{\{v_i, v_j\} \mid v_i, v_j \in V\}$ is a set of edges, where each edge is an unordered pair of vertices. There are many different kind of graphs like path, wheel, prism, etc. For this research, we will specifically study the lobster graph. A lobster graph is a type of tree-like graph that resembles a lobster, hence the name. It consists of a central path (often referred to as the "body" of the lobster) and a series of pendent edges (resembling "claws" of the lobster) connected to the body. The central path is usually a simple path, and each pendent edge is connected to a distinct vertex on the path. Here we will study a lobster graph with three parameters. The precise definition is as follows.

Definition 2.1. A lobster graph $L(p,q,r)$ can be defined as follows:

- The body path consists of p vertices in a sequence, say $v_1, v_2, ..., v_p$ that are connected in that order to create a simple path.
- From each vertex on the body path, we create as many as q vertices (called the hand vertices) that are connected to it. All of these hand vertices are not adjacent to each other.
- From each vertex on the hand layer, we create as many as r vertices (called the finger vertices) that are connected to it. All of these finger vertices are not adjacent to each other.

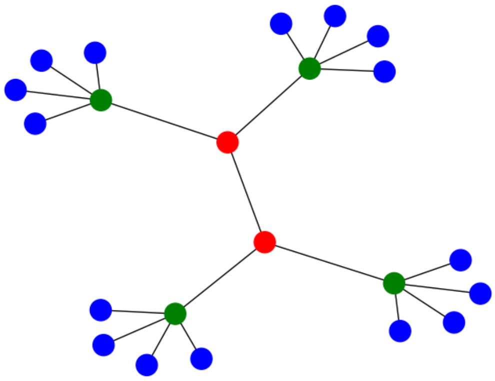

FIGURE 1. $L(2,2,4)$

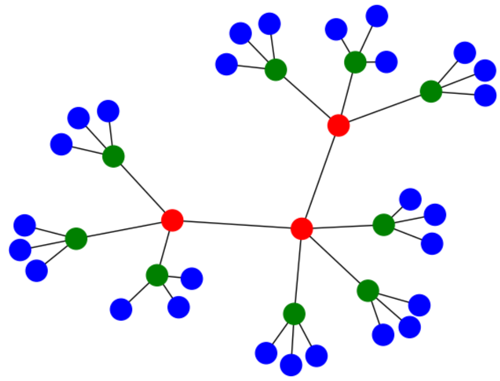

FIGURE 2. $L(3,3,3)$

A labeling of a graph is an assignment of labels to its vertices or edges, or both (total labelling), according to certain rules or constraints. If the maximum label of a graph is k, then we call it a k-labeling. We will study one special case of k-labelling called an *edge irregular reflexive k-labeling* [2].

Definition 2.2. Let $f_e : E(G) \to \{1, 2, ..., k_e\}$ be an edge labeling to positive number and $f_v : V(G) \to \{0, 2, ..., 2k_v\}$ be a vertex labeling to even non-negative numbers. For each $e = \{x, y\} \in E(G)$, define the so called edge irregular reflexive weight as $w(e) = f_v(x) + f_e(\{x, y\}) + f_v(y)$. This labeling is called an edge irregular reflexive k-labeling with $k = max\{k_e, 2k_v\}$, if for every $e, e' \in E(G)$ (these two edges can have an incidence vertex) one has $w(e) \neq w(e')$. The smallest value of k for which such labeling exists is called the *reflexive edge strength* of the graph G, res(G).

To avoid the confusion, we will call the label for one edge as edge-label and the edge irregular reflexive weight of the same edge as edge-weight. Also, if we write the edge-weight as $w = a + b + c$, this means that a and c are the label for incident vertices and b is the edge-label. One general result of the reflexive edge strength of a graph is the lower bound as shown in the following theorem [2].

Theorem 2.1. *For every graph G,*

$$res(G) \geq \begin{cases} \left\lceil \frac{|E(G)|}{3} \right\rceil, & if\ |E(G)| \not\equiv 2,3\ (mod\ 6), \\ \left\lceil \frac{|E(G)|}{3} \right\rceil + 1, & if\ |E(G)| \equiv 2,3\ (mod\ 6). \end{cases}$$

In the next section, we will prove the formula for reflexive edge strength of a general lobster graph. We give a general algorithm to do the *edge irregular reflexive k-labeling* for this lobster graph. Later, we simulate this algorithm and use Python code to illustrate it.

3. MAIN RESULT

Our key finding is a method for labeling the lobster graph that complies with edge irregular reflexive k-labeling requirements. From there, we can deduce the value of reflexive edge strength of lobster graph as outlined on the following theorem.

Theorem 3.1. *Let p, q and r be positive integers with $p,q,r \geq 2$. Let $L(p,q,r)$ be the lobster graph with the total number of edges is $E = p + pq + pqr - 1$. Then*

$$res(L(p,q,r)) = \begin{cases} \left\lfloor \frac{E}{3} \right\rfloor + 1, & if\ \left\lfloor \frac{E}{3} \right\rfloor \equiv 1\ (mod\ 2) \\ \left\lfloor \frac{E}{3} \right\rfloor + (E\ mod\ 3), & if\ \left\lfloor \frac{E}{3} \right\rfloor \equiv 0\ (mod\ 2) \end{cases}$$

Proof. We claim for now that we can construct an algorithm for edge irregular reflexive labelling on lobster graph such that all numbers from range 1 to E is used as its edge-weights and so the value of reflexive edge strength corresponds to the labels on edge or vertices with edge-weights E. Because the maximum edge-weights of this labeling is E, now we look at several different cases as follows.

(1) If $k = \lfloor \frac{E}{3} \rfloor \equiv 1\ (mod\ 2)$, then we divide again to three cases as follows.
- If $E \equiv 0\ (mod\ 3)$, then E can be written as $E = (k-1) + k + (k+1)$
- If $E \equiv 1\ (mod\ 3)$, then E can be written as $E = (k-1) + (k+1) + (k+1)$
- If $E \equiv 2\ (mod\ 3)$, then E can be written as $E = (k+1) + k + (k+1)$

Therefore, it can be concluded that

$$res(L(p,q,r)) = k + 1 = \lfloor E/3 \rfloor + 1$$

(2) If $k = \lfloor \frac{E}{3} \rfloor \equiv 0\ (mod\ 2)$, then, we split into three cases once again as shown below.
- If $E \equiv 0\ (mod\ 3)$, then E can be written as $E = k + k + k$
- If $E \equiv 1\ (mod\ 3)$, then E can be written as $E = k + (k+1) + k$
- If $E \equiv 2\ (mod\ 3)$, then E can be written as $E = k + (k+2) + k$

Therefore, it can be concluded that

$$res(L(p,q,r)) = k + (E\ mod\ 3) = \lfloor E/3 \rfloor + (E\ mod\ 3)$$

We therefore proceed to construct the intended algorithm. We will do the so called backward labeling, which means the starting value of edge irregular reflexive edge-weights that we use will start from E and reduced by 1 for each subsequent edge until it becomes 1. Now, we build the labeling method for the graph as follows.

(1) From the definition, we divide the vertices of the lobster graph into three different sets. The first p vertices will be called layer 1. From each vertex in layer 1, there as many as q vertices that are connected to it. The total of this new pq vertices will be called layer 2. Then from each vertex in layer 2, there are as many as r vertices that are connected to it. The total of this new pqr end vertices will be called layer 3.

(2) We start from the first vertex on layer 1. From the vertex on layer 1, now label the edge and vertex that connect it to layer 2. On the first label, make sure that its edge-weights is E by following the rules as shown in the cases before. The subsequent edge-weights then are reduced by 1 for each iteration.

(3) From this one vertex of layer 2, now label all edges and vertices that connect it to layer 3. The computation of each labels from here will follow the following rules. Assume the edge-weights that we want is w and the label of one vertex that we know is a. Let the intended label of edge and vertex are e and v respectively. We have

$$w = a + e + v \implies w - a = e + v.$$

Choose e and v such that $e, v \geq \lfloor (w-a)/2 \rfloor - 1$. This is to ensure each label does not exceed the reflexive edge strength calculated before. Also, this ensure that the vertices and edges on the body path can always be labeled.

(4) Repeat step 2 and 3 until all hand and finger vertices that are originated from this layer 1's vertex has been labeled.

(5) Moving on to label the edge and vertex that connect it to the next vertex on layer 1. As an exception, if the next vertex is the final vertex on layer 1, then we have to make sure the label for the next vertex is at most $r+1$. This is to ensure that we can give the final vertex on hand layer the label 0 because the edge-weights of this vertex to the finger vertices will be $1, 2, \dots, r$.

(6) Back to step 2. The algorithm ends when all vertices and edges has been labeled.

With this algorithm, we ensure the edge irregular reflexive labeling on lobster graph such that all numbers from range 1 to E are used as its edge-weights. Therefore, the value of reflexive edge strength corresponds to the labels on edge or vertices with edge-weights E. □

4. SIMULATION

For the purpose of ilustrating the edge irregular reflexive labeling on lobster graph, the algorithm from the previous proof will be applied to Python programming language. This will also help us to label a large graph. In Python we utilize the library called `networkx` that contains many function to create and manipulate graph.

Example of the step-by-step procedure to label $L(3,3,3)$. The number of edges of this graph is $E = 38$. The number 38 can be written as $38 = 12 + 14 + 12$. Therefore $res(L(2,2,4)) = 14$. Following the second step of algorithm, first we get the following label.

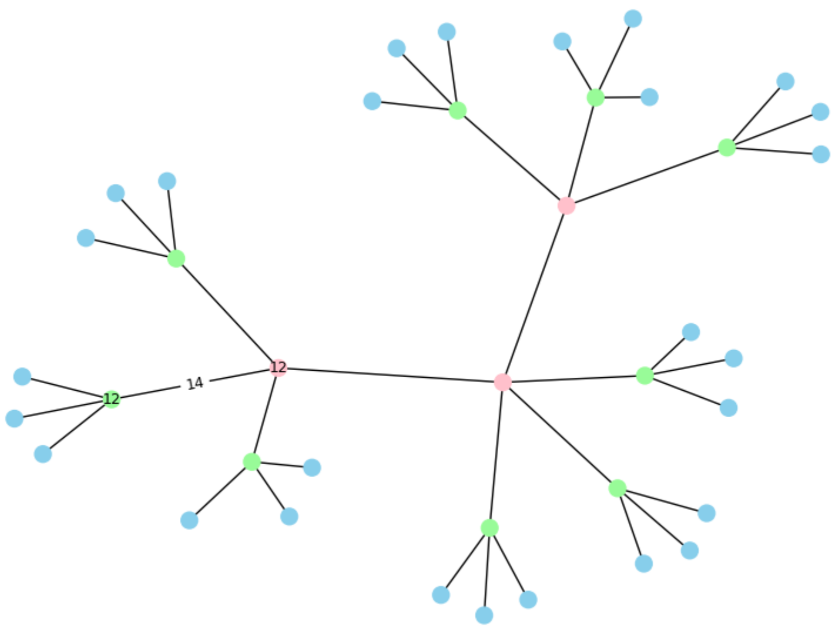

FIGURE 3. First label

Now we move to step 3 to label all the finger vertices and edges that are connected to previous hand vertex according to the rules mentioned in the proof. The rule ensures that the label does not exceed the reflexive edge strength of 14. Actually because the intended edge-weights will become much smaller in the later iteration of labeling, we do not necessarily have to follow this rules (but we have to make sure the labels are still less than the reflexive edge strength).

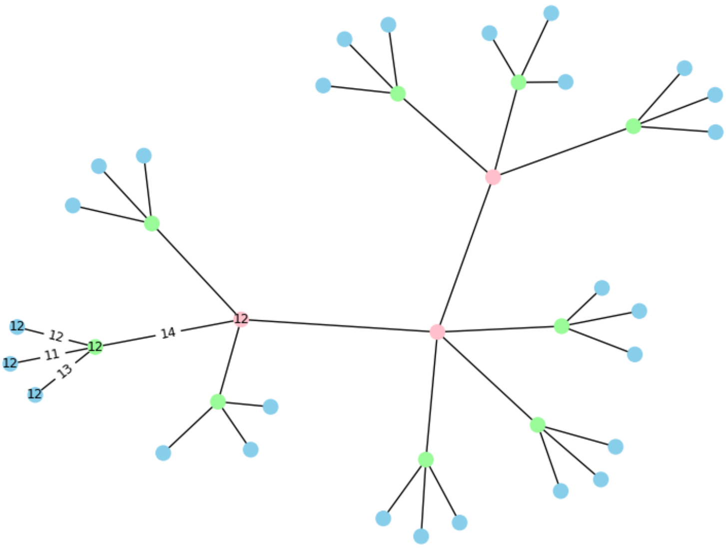

FIGURE 4. Step 3 of algorithm

Next, we repeat the previous two steps to all vertices and edges of the hand and finger layers that are originated from this first body vertex.

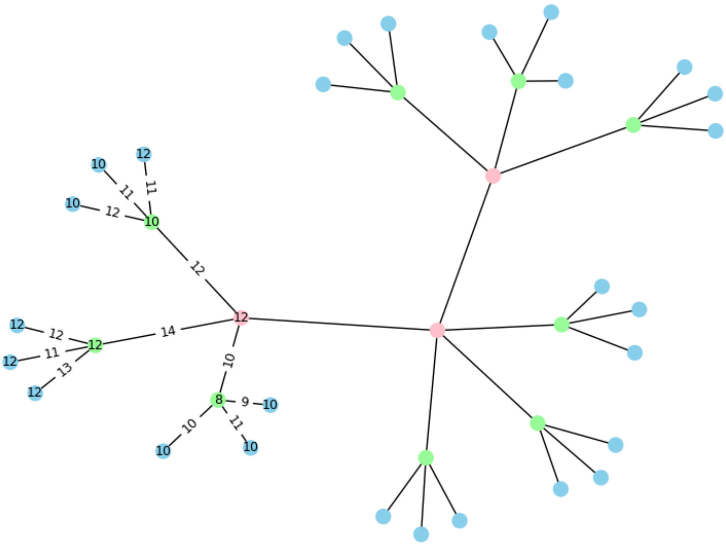

FIGURE 5. Step 4 of algorithm

The vertex and edge that join this initial vertex to the following vertex on the body component are then labeled. To label the final vertex on this body path at most $r+1$, we must closely abide by the rules outlined in the algorithm of the proof.

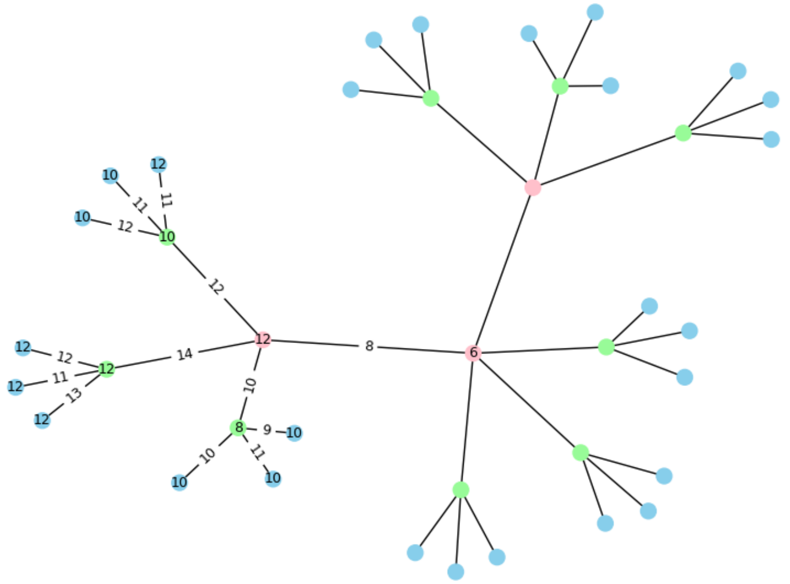

FIGURE 6. Step 5 of algorithm

After we repeat all the previous process, we got the following final label for $L(3,3,3)$ as follows.

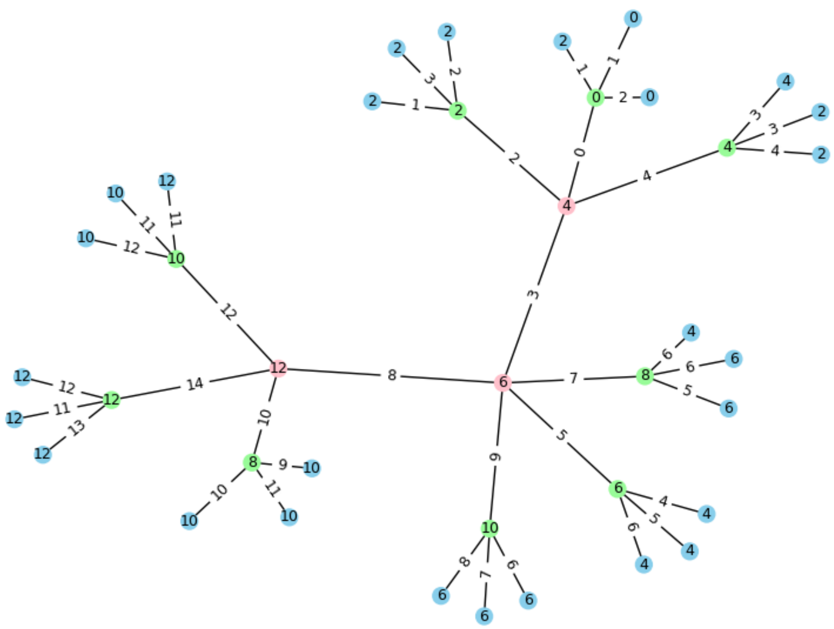

FIGURE 7. Edge Irregular Reflexive Labeling on $L(3,3,3)$

Other example of edge irregular reflexive labeling on various lobster graph are as follows. All of this is generated by Python code that can be seen at https://github.com/verreld7/res-lobster.

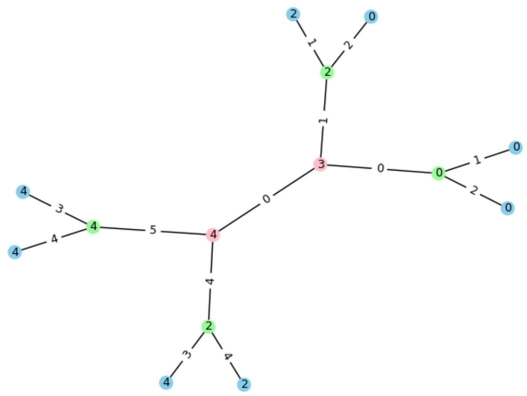

FIGURE 8. Label on $L(2,2,2)$

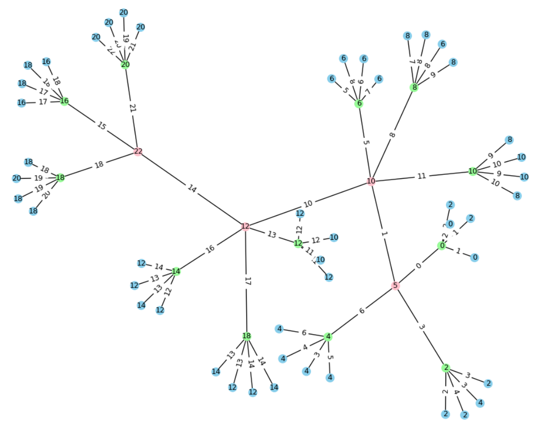

FIGURE 9. Label on $L(4,3,4)$

5. CONCLUSION

In this paper, we examine the edge irregular reflexive k-labeling on lobster graph. We propose an algorithm to achieve the edge irregular reflexive labeling of lobster graph such that k is minimum. From there we get the general formula for the value of k, also called the reflexive edge strength of the graph. The proposed algorithm then is implemented in Python code to confirm the theorem and simulate the labeling on any lobster graph.

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